The limit-2 case of a second-order differential system

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Abstract

A technique is developed for identifying the system

$$M [ \Psi ] = \left( \begin{array}{cc}
- \frac{d}{dx} \left( p \frac{d}{dx} \right) + q_1 & q_2 \\
q_2 & - \frac{d}{dx} \left( r \frac{d}{dx} \right) + q_1
\end{array} \right) \quad \Psi = \lambda \Psi, \lambda \in \mathbb{C}
$$

to be in the limit-2 case at infinity

Key words: Limit-2 case at infinity, Hilbert space, spectral theory, Lebesgue integrable, linear manifold, bilinear form

I. Introduction

Let $M$ denote the formally symmetric second-order vector-matrix differential expression given by

$$M [ \Psi ] = \left( \begin{array}{cc}
- \frac{d}{dx} \left( p \frac{d}{dx} \right) + q_1 & q_2 \\
q_2 & - \frac{d}{dx} \left( r \frac{d}{dx} \right) + q_1
\end{array} \right) \quad \Psi
$$

(1.1)

$\Psi$ being a complex-valued vector function $\Psi = (f, g)^T$, suitably differentiable on the interval $(0, \infty)$ and where the coefficients $p, r$ and $q_j$ $(j = 1, 2, 3)$ satisfy the following conditions:
(i) \( p(x), r(x) \) are real-valued and positive for all \( x \) on \((0, \infty)\) and are absolutely continuous on all compact sub-intervals of \((0, \infty)\).

(ii) \( q_j (j = 1, 2, 3) \) are real-valued and continuous on \((0, \infty)\).

The Hilbert space \( H \) in which the spectral theory of \( M \) is developed is that of complex-valued vector-functions \( \Psi = (f, g) \) such that

\[
\int_0^\infty \left\{ |f|^2 + |g|^2 \right\} \, dx < \infty
\]  

or, equivalently, each of \( \text{Re}(f), \text{Re}(g), \text{Im}(f), \text{Im}(g) \) is square-integrable on \((0, \infty)\); we express these by writing \( \text{Re}(f), \text{Re}(g), \text{Im}(f), \text{Im}(g) \in L^2(0, \infty) \). The inner product of two vectors \( \Psi = (f, g) \) and \( \Phi = (\Psi', \Phi') \) is defined by

\[
(\Psi, \Phi) = \int_0^\infty (f\overline{\mu} + g\overline{\nu}) \, dx.
\]

It is known [See Chakravarty\(^1\), Sengupta\(^2\), Naimark\(^3\) (§ 17.5 VII) and Glazman\(^4\) (Ch. I. § 13)] that the differential system

\[
M[\Psi] = \lambda \Psi, \quad \text{Im}\lambda \neq 0
\]  

possesses at least two and at most four linearly independent solutions on \((0, \infty)\) which lie in \( H \). \( M[\cdot] \) is said to be in the limit-S case at infinity if the differential system (1.3) has exactly \( S \) number of linearly independent solutions in \( H \). Given \( p, r, q_1, q_2, q_3 \) the number \( S \) is independent of \( \lambda \), as long as \( \text{im}\lambda \neq 0 \). The idea of this paper is to establish a general set of sufficient conditions on the coefficients \( p, r, q_1, q_2, q_3 \) so that \( M[\cdot] \) is in the limit-2 case at infinity. Several methods have been used for investigating the limit-2 case for the system (1.3) or for one similar to it. In 1954, Lidskii\(^5\) showed that the system

\[
- Y'' + Q Y = \lambda Y, \text{Im}\lambda \neq 0
\]  

possesses \( k \) number of linearly independent square-integrable solutions on \((0, \infty)\) provided the square hermitian matrix \( Q(x) \) of order \( k \) satisfies

\[
(Q(x) \, h, h) \geq - N(x) \| h \|^2
\]

for any constant \( k \)-vector \( h \), where the positive continuous function \( N(x) \) satisfies

(i) \[ \int_0^\infty \left[ N(x) \right]^{-1/2} \, dx \text{ diverges} \]

and, either

(ii) \( N(x) \) is monotone

or,

(iii) \( N(x) \) is differentiable and \( \lim_{x \to \infty} \frac{N'(x)}{[N(x)]^{3/2}} < \infty \).
Seif's result\(^6\) can be derived from Lidskii's result by putting \(k=1\). Chakravarty\(^7\) (Th. III) proved in a different way that the system

\[
M_1[\psi] = \begin{pmatrix}
q_1 & -\frac{d^2}{dx^2} + q_2 \\
-\frac{d^2}{dx^2} + q_2 & q_3
\end{pmatrix}
\phi = \lambda \psi
\]

(1.6)

is in the limit-2 case at infinity if \(q_1, q_2, q_3\) are all \(O(x^2)\) as \(x \to \infty\). Anderson\(^8\) discussed the system

\[
\psi^{(2n+1)} + Q \psi = \lambda \psi
\]

(1.7)

where \(Q\) is a \(k \times k\) matrix of real measurable functions which are Lebesgue integrable on compact sub-intervals of \((0, \infty)\) and \(\Psi\) is a \(k\)-vector, and extended the results of Lidskii\(^5\) to the case when the system (1.7) possesses the minimum number \((viz. nk)\) of square-integrable solutions on \((0, \infty)\). The method applied by Anderson is similar to that applied by Hinton\(^9\) to the corresponding scalar equation. In particular, if \(n = 1, k = 2\) Anderson proved that [Th. 2.4], the system

\[
\psi'' + \begin{pmatrix}
q_1 & q_2 \\
q_2 & q_3
\end{pmatrix} \psi = t \psi
\]

is in the limit-2 case at infinity if \(q_1, q_3, |q_2| \leq N(x)\) for \(N(x)\) as in (1.5). Following Titchmarsh\(^10\) (Th. 2.20) Bhagat and Guma\(^11\) (§ 5) pointed out that the system (1.3) with \(p=r=1\) is in the limit-2 case at infinity, if \(q_2=0(1)\) and \(q_1, q_3 \geq -N(x)\) is a positive, continuous non-decreasing function of \(x\) satisfying condition (i) of (1.3). A complete analysis of the system (1.6) has been made by Eastham\(^12\) when \(q_j's, j=1,2,3\) are multiples of powers of \(x\), giving conditions under which \(S=2\) or \(S=3\) or 4. In this connection mention should also be made of the papers by Titchmarsh\(^13\), Shaw and Bhagat\(^14\), Sengupta\(^15,16\), Eastham\(^17\) and Everett\(^18,19\).

In this paper, we present a simpler method to establish that the system (1.3) is in the limit-2 case at infinity under suitable conditions imposed on the coefficients \(p, r, q_1, q_2, q_3\) which will include the cases mentioned earlier. The method employed is based on an extension of a technique given in Levinson\(^21\) or Coddington and Levinson\(^22\) (Th. 2.4 Ch. 9, Sec. 2). The result obtained is given in the following theorem:

**Theorem:** Let \(N(x)\) be a positive, absolutely continuous and non-decreasing function of \(x\) such that
(i) \[ \int_0^\infty \left[ PN \right]^{-1/2} \, dx \text{ diverges, } P = \max(p, r) \]  
(ii) \[ \lim \sup_{x \to \infty} N' \sqrt{|p| N^3} \text{ and } \lim \sup_{x \to \infty} N' \sqrt{(r/ N^3) } \text{ exist finitely} \] 
and moreover,

(iii) \[ q_1(x) \geq -k_1 \, N(x), \quad q_2(x) \geq -k_1 \, N(x) \text{ and } q_2(x) \leq k_2 \, N(x) \quad (1.10) \]

\( k_1, k_2, k_3 \) are all finite positive constants) hold for all sufficiently large values of \( x \).

Then \( M[\cdot] \) is in the limit-2 case at infinity.

The proof is given in the following section. In proving the theorem we extract a function

\[ W(x) = \int_0^x \left[ (\theta^T \, R \, \theta'/N) \right] \, dx \quad \left[ R = \begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix} \right], \]

from the equation

\[ \int_0^x (\theta^T \, M[\theta]/N) \, dx = i \int_0^x (\theta''/N) \, dx, \]

converging to a finite limit as \( x \to \infty \), which later produces \( (R \theta'/\sqrt{PN}) \in H \) for all \( \theta \in D \) [See section 2 for definition of \( D \)]. Finally the theorem follows on utilising the last result along with (1.8) and (2.1).

2. Proof of the theorem

We introduce a linear manifold \( D \) as follows:

A vector-valued function \( \Psi = \begin{pmatrix} f \\ g \end{pmatrix} \) is in \( D \) if and only if

(i) \( \Psi \in H \)
(ii) \( f', g' \) are absolutely continuous on \( (0, \infty) \)
(iii) \( M[\Psi] \in H \)

For \( \Psi = \begin{pmatrix} f \\ g \end{pmatrix} \), \( \Phi = \begin{pmatrix} \psi' \\ \psi \end{pmatrix} \in D \), it is known from Green's formula that

\[ \int_0^\infty \bar{\Phi}^T \, M[\Psi] \, dx - \int_0^\infty \bar{\Psi}^T \, M[\Phi] \, dx = \left\{ p \left( \tilde{f} \Psi' - f' \bar{\Psi} \right) + r \left( \tilde{g} \Psi' - g' \bar{\Psi} \right) \right\} \]

and the bilinear form

\[ [\Psi \, \Phi] = p \left( \tilde{f} \Psi' - f' \bar{\Psi} \right) + r \left( \tilde{g} \Psi' - g' \bar{\Psi} \right) \text{ tends to a finite limit as } x \to \infty \quad (2.1) \]

and that

\[ \lim_{\imath \to \infty} [\Psi \, \Phi] = 0 \quad (2.2) \]
for all $\Psi, \Phi \in D$ if and only if $M$ is in the limit-2 case at infinity [See Sengupta\textsuperscript{2} Th. 6.2; Naimark\textsuperscript{1} § 18.3 lemma].

Since the number of $L^2$-solutions of the system (1.3) remains unchanged as long as $\text{im} \lambda \neq 0$, we start to prove the theorem by choosing $\lambda = i$ in it.

Let $\Psi = \begin{pmatrix} f \\ g \end{pmatrix} \in D$ be a solution of $M \Psi = i \Psi$ satisfying the initial conditions

$$
\begin{align*}
&f(0) = \alpha, g(0) = \beta \\
&p(a)f'(a) = \gamma, r(a)g'(a) = \delta
\end{align*}
$$

$a, \beta, \gamma, \delta$ are finite complex constants [For existence of the initial conditions, see Sengupta\textsuperscript{2} Th. 3.1].

Multiply both sides of $M \Psi = i \Psi$ by $(\Psi^* / N)$, integrate between $a$ and $x$, and then integrating the right-hand side by parts, we get

$$
- \int_a^x q \frac{f'}{N} + q_2 (f g + f^2) + p g^2 \, dx + \int_a^x \left| f \right|^2 + \left| g \right|^2 \, dx = - \int_a^x \frac{(pf')f + (rg')g}{N} \, dx
$$

Taking real parts from both the sides,

$$
- \int_a^x q \frac{|f|^2 + 2q_2 (f g + f^2) + q_3 |g|^2}{N} \, dx = - \frac{p \left( f_1 f_1' + f_2 f_2' + r (g_1 g_1' + g_2 g_2') \right)}{N} \left| \Psi \right|^2 + \\
+ \int_a^x \frac{|f'|^2 + r |g'|^2}{N} \, dx - \int_a^x \frac{p \left( f_1 f_1' + f_2 f_2' + r (g_1 g_1' + g_2 g_2') \right)}{N^2} \, dx
$$

then by condition (1.10) l.h.s. satisfies the inequality

$$
- \int_a^x q \frac{|f|^2 + 2q_2 (f g + f^2) + q_3 |g|^2}{N} \, dx \leq - \int_a^x \frac{q_1 |f|^2 + q_3 |g|^2}{N} \, dx + \\
+ \frac{2}{N} \int_a^x |g_2| \left( f_1 g_1 + f_2 g_2 \right) \, dx < k_1 \int_a^x |f|^2 \, dx + k_1 \int_a^x |g|^2 \, dx + 2k_2 \int_a^x \left| f_1 g_1 + f_2 g_2 \right| \, dx
$$

Hence there exists a constant $K$ such that

$$
K > \frac{p \left( f_1 f_1' + f_2 f_2' + r (g_1 g_1' + g_2 g_2') \right)}{N} + \int_a^x \frac{|f'|^2 + r |g'|^2}{N} \, dx - \\
- \int_a^x \frac{p \left( f_1 f_1' + f_2 f_2' + r (g_1 g_1' + g_2 g_2') \right)}{N^2} \, dx (\Psi x)
$$

(2.3)
Now it is to be proved that if the solution $\Psi \in D$ then the integral

$$\int_{a}^{b} \frac{p|f'|^2 + r|g'|^2}{N} \, dx$$

converges. For, suppose conversely that this integral diverges, then the function

$$W(x) = \int_{a}^{x} \frac{p|f'|^2 + r|g'|^2}{N} \, dx$$

is positive, monotonically increasing and tends to $+\infty$ as $x \to \infty$. Using condition (1.9) and then the Cauchy-Schwartz inequality results in

$$\left| \int_{a}^{x} \frac{p(f_1 f_1' + f_2 f_2') + r(g_1 g_1' + g_2 g_2')}{N^2} \, dx \right| \leq K_1 \left( \int_{a}^{x} \frac{p(f_1^2 + f_2^2) + r(g_1^2 + g_2^2)}{N} \, dx \right)^{1/2} \left( \int_{a}^{x} \frac{p|f'|^2 + r|g'|^2}{N} \, dx \right)^{1/2}$$

Applying these results in (2.3), we find that

$$K > W(x) - \frac{p(f_1 f_1' + f_2 f_2') + r(g_1 g_1' + g_2 g_2')}{N} = K_2 \sqrt{W(x)}$$

Since $W(x) \to \infty$ as $x \to \infty$, the last inequality can hold only if

$$\frac{p(f_1 f_1' + f_2 f_2') + r(g_1 g_1' + g_2 g_2')}{N} > 1/2 \, W(x)$$

for all sufficiently large $x$. As $p, r$ and $N$ are positive it appears from the above inequality that at least one of the pairs $f_1, f_1'; f_2, f_2'; g_1, g_1'; g_2, g_2'$ is of the same sign for large $x$. In this situation at least one of the four integrals

$$\int_{a}^{x} \frac{p|f'|^2 + r|g'|^2}{N} \, dx$$

is positive, monotonically increasing and tends to $+\infty$ as $x \to \infty$. Using condition (1.9) and then the Cauchy-Schwartz inequality results in
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\[ \int f_1^2 \, d\mathbf{x}, \int f_2^2 \, d\mathbf{x}, \int g_1^2 \, d\mathbf{x}, \int g_2^2 \, d\mathbf{x} \]

fails to exist and this contradicts the fact that \( \Psi \in D \). Thus, \( W(x) \) remains finite for \( \Psi \in D \) and that

\[ \int_{\mathbb{D}} \frac{p|f'|^2 + r|g'|^2}{N} \, d\mathbf{x} < \infty. \]

it then follows

\[ \sqrt{p/N} |f'|, \sqrt{r/N} |g'| \in L^2(0, \infty) \]

and consequently

\[ \frac{p|f'|}{\sqrt{pN}} \leq \frac{p|f'|}{\sqrt{pN}} = \sqrt{p/N} |f'| \in L^2(0, \infty), \quad (2.4a) \]

\([P = \max (p, r)] \) and likewise

\[ \frac{r|g'|}{\sqrt{pN}} \in L^2(0, \infty) \quad (2.4b) \]

for all \( \Psi \in D \).

Utilizing the results obtained we now show that \( \lim_{\text{lim}} \Psi \Phi = 0 \) for any solution \( \Psi, \Phi \in D \).

From (2.1) we find

\[ \int_{\mathbb{D}} \frac{|\Psi \Phi|}{\sqrt{PN}} \, d\mathbf{x} = \int_{\mathbb{D}} \frac{|p\mathbf{u}' - pf' \mathbf{u} + r\mathbf{v}' - rg' \mathbf{v}|}{\sqrt{PN}} \, d\mathbf{x} \]

\[ \leq \int_{\mathbb{D}} \frac{p|f||\mathbf{u}'| + p|f'||\mathbf{u}' + r|g||\mathbf{v}'| + r|g'||\mathbf{v}|}{\sqrt{PN}} \, d\mathbf{x} \]

The integral on the right side converges as \( x \) tending to infinity following the result (2.4) for \( \Psi, \Phi \in D \) and consequently

\[ \int_{\mathbb{D}} \frac{|\Psi \Phi|}{\sqrt{PN}} \, d\mathbf{x} \]
converges. Now, if \( \lim_{t \to \infty} [\Psi \Phi] = k \), a finite limit (\( \neq 0 \)), we can find an \( X_k \in (0, \infty) \), depending on \( k \) such that

\[
1/2 |k| < |[\Psi \Phi]| < 3/2 |k|
\]

hold for all \( x > X_k \). Then

\[
\int_a^b \frac{|[\Psi \Phi]|}{\sqrt{PN}} \, dx = (\int_a^b + \int_b^c) \frac{|[\Psi \Phi]|}{\sqrt{PN}} \, dx \geq I_1 + 1/2 |k| \int_a^b \sqrt{PN} \to -\infty,
\]

as \( X \to \infty \) contradictory to (2.5) and the desired result \( \lim_{t \to \infty} [\Psi \Phi] = 0 \) is achieved, which ensures the system \( M[\cdot] \) to be in the limit-2 case at infinity.

**Remark 1.** Some discrepancy is found in between the papers of Gadamsi-Maho 23 and Eastham-Gould 24. In Theorem 3 24 the authors tried to apply the techniques of Titchmarsh 11 and Everitt 20 in proving the system (1.3) to be in the limit-2 case at infinity; the conditions taken there were

(i) \( 0 < p, r \leq k x^q \)  
(ii) \( q_1, q_1 \geq -k_1 x^a, q_2 \geq -k_2 x^r \)

with \( \alpha + \beta \leq 2, \alpha \geq 0, \beta - \alpha \leq 2 \gamma \leq \alpha \) and \( k, k_1, k_2 \) all positive finite constants; whereas in Theorem 1 (ii), 324 it was proved that the system (1.3), (with \( p = r = 1 \)) is in the limit-3 case at infinity provided

\[
q_1 = a \leq q_2, q_2 = b \geq q_3, a \geq 0, b \geq 0, ab < 1
\]

where \( q_2 \) be no where zero in some interval \([X, \infty), X \geq 0 \) and \( q_2^{-1/4}(q_2^{-1/4})^\gamma \in L^2(X, \infty) \).

As for an example, if we take \( p = r = 1 \) and \( q_1 = q_3 = 1/2 \, x^3, q_2 = x^3, x \in (0, \infty) \), then following Eastham-Gould 24 the system (1.3) turns out to be in the limit-3 case at infinity though the coefficients \( p, r, q_1, q_2, q_3 \) satisfy the conditions of Gadamsi-Maho 23.

**Remark 2.** It appears from the previous results (except Gadamsi-Maho 23) especially Anderson 8, Th.2.4 and the theorem of the present paper that for a system of the type (1.3), belonging to the limit-2 case at infinity, \( q_2 \) should satisfy

\[
|q_2(x)| \leq KN(x)
\]

along with other restrictions on \( q_1 \) and \( q_3 \).

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