On approximate strong and approximate uniform differentiability

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Abstract

In this paper we introduce the definitions of approximate strong and approximate uniform differentiability and prove that a function having finite approximate derivative on an interval $I$ is approximately uniformly differentiable at a point if and only if it is approximately strongly differentiable there. We also introduce the definition of essentially $AC$ [BV] functions and prove that a measurable BVG function on a set with finite measure is essentially $AC$.

Key words: Uniform differentiability, strong differentiability, approximate uniform differentiability.

1. Introduction

Let $f$ be a real-valued function on an open interval $I$. Lahiri\(^*\) introduced the following definition.

**Definition 1.1:** Let $f' (x)$ exist finitely at each point of $I$. A point $a \in I$ is said to be a point of uniform differentiability of $f$ if for every $\varepsilon > 0$ there is a neighbourhood $\triangle_a (\delta) = (a - \delta, a + \delta) \subset I$ and a positive number $P$ such that

$$\left| \frac{f(x + h) - f(x)}{h} - f'(x) \right| < \varepsilon$$

for all $x \in \triangle_a (\delta)$ and $x + h \in I$ whenever $0 < |h| \leq P$. Otherwise $a$ is said to be a point of non-uniform differentiability of $f$.

Lahiri proved the following results.

**Theorem 1.1:** Let $f' (x)$ exist finitely at each point of $I$. A point $a \in I$ is a point of uniform differentiability of $f$ iff $f'$ is continuous at $a$.

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Theorem 1.2: Let \( f'(x) \) exist finitely at each point of \( I \). Then the set of the points of non-uniform differentiability of \( f \) in \( I \) is of first category and is an \( F_\sigma \).

Esser and Shisha\(^3\) introduced the following definition.

**Definition 1.2:** \( f \) is said to be strongly differentiable at \( a \in I \) if the double limit

\[
\lim_{(x, y) \to (a, a)} \frac{f(x) - f(y)}{x - y}
\]

exists finitely. This limit whenever exists is denoted by \( f^*(a) \) and is called the strong derivative of \( f \) at \( a \).

Esser and Shisha proved the following result.

**Theorem 1.3:** Let \( f'(x) \) exist finitely at each point of \( I \). Then \( f \) is strongly differentiable at \( a \in I \) iff \( f' \) is continuous at \( a \).

From Theorems 1.1 and 1.3 we immediately derive the following result.

**Theorem 1.4:** Let \( f'(x) \) exist finitely on \( I \). A point \( a \in I \) is a point of uniform differentiability of \( f \) iff \( f \) is strongly differentiable at \( a \).

In Section 2, we prove some results on strong and uniform differentiability. In Section 3, we introduce the definitions of approximate strong and approximate uniform differentiability and extend the above results in the light of these new definitions and prove some other results. In Section 4, we introduce the definition of essentially \( AC \) \([BV]\) functions and prove, among other results, that if \( f \) is \( BV \) and measurable on a set \( E \) with \( mE < +\infty \), then \( f \) is essentially \( AC \) on \( E \).

2. Results on strong and uniform differentiability

**Definition 2.1:** Let \( f \) and \( g \) be two real-valued functions on \( [a, b] \). Suppose that for every \( a \in (a, b) \) there is a set \( S_a \subset (a, b) \) having \( a \) as two-sided limiting point such that

\[
\lim_{x \to a \atop x \neq a} \frac{f(x) - f(a)}{x - a}
\]

exists and equals \( g(a) \).

At the points \( a \) and \( b \) necessary modifications are made. In this case we call \( g \) a derived function of \( f \) on \( [a, b] \).

**Note 2.1:** If the approximate derivative \( f'_a(x) \) exists finitely on \( [a, b] \), then \( f'_a \) is a derived function of \( f \) on \( [a, b] \).

**Theorem 2.1:** Let \( f \) be a real-valued bounded function on \( [a, b] \) and let \( f \) attain its bounds on \( [a, b] \). If \( f \) possesses a derived function \( g \) on \( (a, b) \) and \( f(a) = f(b) \), then there is a point \( c \) in \( (a, b) \) such that \( g(c) = 0 \).
**Proof**: If \( f(x) = f(a) = f(b) \) for all \( x \in [a, b] \), then \( g(x) = 0 \) on \((a, b)\). Suppose that there is a point \( \zeta \) in \((a, b)\) with \( f(\zeta) > f(a) = f(b) \). Denote by \( M \) the l.u.b. of \( f \) on \([a, b]\). Then there is a point \( c \) in \([a, b]\) such that \( M = f(c) \). Since \( M > f(a) = f(b) \) we have \( a < c < b \) and

\[
 f(x) - f(c) \leq 0 \quad \text{for all } x \in [a, b].
\]

(2.1)

For any \( x \) in \((c, b)\) we get

\[
 \frac{f(x) - f(c)}{x - c} \leq 0.
\]

Letting \( x \to c \) over the set \( S_c \) (see def. 2.1), we obtain

\[
 g(c) \leq 0.
\]

(2.2)

Again for any \( x \) in \((a, c)\) we get from (2.1)

\[
 \frac{f(x) - f(c)}{x - c} \geq 0.
\]

Letting \( x \to c \) over the set \( S_c \) we have

\[
 g(c) \geq 0.
\]

(2.3)

Combining (2.2) and (2.3) we obtain \( g(c) = 0 \).

If there is a point \( \eta \) in \((a, b)\) with \( f(\eta) < f(a) = f(b) \), then proceeding as above and using the property of g.l.b. of \( f \) on \([a, b]\) we deduce that \( g(c) = 0 \) for some point \( c \) in \((a, b)\). This completes the proof.

**Corollary 2.1.1**: Let \( f \) be continuous on \([a, b]\) and possess a derived function \( g \) on \((a, b)\). Then there is a point \( c \) in \((a, b)\) such that

\[
 f(b) - f(a) = (b - a) g(c).
\]

**Corollary 2.1.2**: Let \( f \) be continuous on \([a, b]\) and let \( f'_a(x) \) exist finitely at each point of \((a, b)\). Then there is a point \( c \) in \((a, b)\) such that

\[
 f(b) - f(a) = (b - a) f'_a(c).
\]

**Theorem 2.2**: Let \( f \) be continuous on the open interval \( I \) and let \( f \) possess a derived function \( g \) on \( I \). Then \( f \) is strongly differentiable at \( a \in I \) iff \( g \) is continuous at \( a \).

**Proof**: First suppose that \( g \) is continuous at \( a \). Choose any \( \varepsilon > 0 \). Then there is a \( \delta > 0 \) such that \( (a - \delta, a + \delta) \subseteq I \) and

\[
 |g(x) - g(a)| < \varepsilon \quad \text{for all } x \in (a - \delta, a + \delta).
\]

(2.4)
Take any two points $x, y$ ($x \neq y$) in $(a - \delta, a + \delta)$. By Corollary 2.2.1, there is a point $\xi$ lying between $x$ and $y$ such that
\[ f(x) - f(y) = (x - y) g(\xi). \]
Clearly $\xi \in (a - \delta, a + \delta)$. So using (2.4) we have
\[ |\frac{f(x) - f(y)}{x - y} - g(a)| = |g(\xi) - g(a)| < \varepsilon. \]
This gives that $f^*(a)$ exists and $f^*(a) = g(a)$.

Next, let $f$ be strongly differentiable at $a$. Then $f'(a)$ exists and $f'(a) = g(a) = f^*(a)$. Choose any $\varepsilon > 0$. There is a $\delta > 0$ such that $(a - \delta, a + \delta) \subset I$ and
\[ |\frac{f(x) - f(y)}{x - y} - g(a)| < \varepsilon \quad (2.5) \]
for all $x, y$ ($x \neq y$) in $(a - \delta, a + \delta)$. Keeping $x$ fixed and letting $y \to x$ over set $S$, (see def. 2.1) in (2.5) we get
\[ |g(x) - g(a)| \leq \varepsilon. \]
This gives that $g$ is continuous at $a$.

**Corollary 2.2.1**: Let $f$ be continuous on $[a, b]$ and possess a derived function $g$ on $[a, b]$. If $g$ is continuous on $[a, b]$, then $f'(x)$ exists on $[a, b]$ and $f'(x) = g(x)$ on $[a, b]$.

**Corollary 2.2.2**: Let $f$ be continuous on $[a, b]$ and let $f'(a)$ exist finitely at each point of $[a, b]$. Then $f$ is strongly differentiable at each point of continuity of $f'(a)$.

**Definition 2.2**: Let $f$ be a real-valued function on the set $E$ and let $a \in E$ and be a limiting point of $E$. If the ratio
\[ f(x) - f(a) \]
tends to a limit (finite or infinite) as $x$ tends to $a$ over the set $E$, we denote this limit by $f'_E(a)$ and call it the derivative of $f$ at $a$ relative to the set $E$.

**Definition 2.3**: Let $f$ be a real-valued function on the set $E$ which is dense in itself and let $f'_E(x)$ exist finitely at each point of $E$. $f$ is said to be uniformly differentiable on $E$ if for any $\varepsilon > 0$ there is a positive number $\delta$ such that
\[ \left| \frac{f(y) - f(x)}{y - x} - f'_E(x) \right| < \varepsilon \]
for all $x, y$ ($x \neq y$) in $E$ whenever $|x - y| < \delta$.

**Theorem 2.3**: Let $f$ be a real-valued function measurable on the set $E$ which is dense in itself and $mE < +\infty$ and let $f'_E(x)$ exist finitely at each point of $E$. Then given
any \( \varepsilon > 0 \) there is a perfect set \( C \subset E \) such that \( m(E - c) < \varepsilon \) and \( f \) is uniformly differentiable on \( C \).

**Proof:** Clearly \( f \) is continuous on \( E \). By Theorem 4.3 ([5], Ch. IV, p. 113) \( f'_E \) is measurable on \( E \). Let \( \varepsilon \) be any given positive number. Then by Lusin's Theorem ([5], Theorem 7.1, Ch. III, p. 72) there is a closed set \( F \subset E \) such that \( m(E - F) < \frac{1}{2} \varepsilon \) and \( f'_E \) is continuous on \( F \).

For any two positive integers \( K \) and \( n \) denote by \( E_n(K) \) the set of all points \( x \in F \) such that

\[
\left| \frac{f(y) - f(x)}{y - x} - f'_E(x) \right| \leq 2^{-K}
\]

whenever \( y \in E, y \neq x \) and \( |y - x| \leq 1/n \). Then clearly for each \( K \),

\[
E_1(K) \subset E_n(K) \subset E_2(K) \ldots \text{ and } F = \bigcup_{n=1}^{\infty} E_n(K).
\]

Let \( x \in E_n(K) \) and \( y \in E, y \neq x \) and \( |y - x| \leq 1/2n \). If \( x \in E_n(K) \), then clearly (2.6) holds. Suppose that \( x \notin E_n(K) \). Then we can choose a sequence \( \{x_v\} \) of points in \( E_n(K) \) such that \( x_v \to x \) as \( v \to +\infty \). We may suppose that \( |x - x_v| \leq 1/2n \) and \( x_v \neq y \) for \( v = 1, 2, 3, \ldots \). Then \( |y - x_v| \leq |y - x| + |x - x_v| \leq 1/n \).

So

\[
\left| \frac{f(y) - f(x_v)}{y - x_v} - f'_E(x_v) \right| \leq 2^{-K} \quad (v = 1, 2, 3, \ldots).
\]

Since \( f \) and \( f'_E \) are continuous on \( F \), letting \( v \to +\infty \) we see that (2.6) holds. Thus

\[
\left| \frac{f(y) - f(x)}{y - x} - f'_E(x) \right| \leq 2^{-K}
\]

(2.7)

for \( x \in E_n(K) \) and \( y \in E, y \neq x \) whenever \( |x - y| \leq 1/2n \). We have for each \( K \)

\[
E_1(K) \subset E_2(K) \subset E_3(K) \subset \ldots \text{ and } F = \bigcup_{n=1}^{\infty} E_n(K).
\]

So, for each \( K \), there is a positive integer \( n_K \) such that

\[
m(F - E_{n_K}(K)) < \varepsilon/2^{K+1}.
\]

Take \( B_K = E_{n_K}(K) \) and \( B = \bigcap_{K=1}^{\infty} B_K \).

Then \( B \) is closed and \( m(F - B) \leq \sum_{K=1}^{\infty} m(F - B_K) < \frac{1}{3} \varepsilon \). Denote by \( C \) the set of all condensation points of \( B \). Then \( C \) is a perfect set, \( C \subset B \subset F \subset E \) and

\[
m(E - c) \leq m(E - F) + m(F - B) + m(B - c) < \varepsilon.
\]
Let $\eta$ be any positive number. Choose positive integer $K$ such that $2^{-K} < \eta$.

Take $\delta = 1/2n_K$. Let $x, y \in C$, $y \neq x$ and $|x - y| < \delta$. Then $x, y \in E_{n_K}(k)$ and $|x - y| < 1/2n_K$. So by (2.7)

$$\left| \frac{f(y) - f(x)}{y - x} - f'_E(x) \right| \leq 2^{-K} < \eta.$$ 

This gives that $f$ is uniformly differentiable on $C$.

**Theorem 2.4:** Let $f$ be a real-valued function measurable on the set $E$ which is dense in itself and $mE < \frac{1}{2} \infty$. Suppose that $f'_E(x)$ exists finitely at each point of $E$. Then given any $\varepsilon > 0$, there is a perfect set $C \subset E$ with $m(E - C) < \varepsilon$ and a real-valued function $g$ on the real line having the following properties: (i) $g$ possesses continuous derivative on the real line, (ii) $g(x) = f(x)$ and $g'(x) = f'_E(x)$ for all $x \in C$.

**Proof:** Choose any $\varepsilon > 0$. Then by Theorem 2.3 there exists a bounded perfect set $C \subset E$ with $m(E - C) < \varepsilon$ such that $f$ is uniformly differentiable on $C$. Clearly $f$ and $f'_E$ are continuous on $C$.

Denote by $a$ and $b$ the g.l.b. and l.u.b. of the set $C$ and let $G = [a, b] - C$. Then $G$ is an open set and so can be expressed in the form $G = U \cup (a_n, b_n)_n$, where the open intervals $(a_n, b_n), (a_{n+1}, b_{n+1}), \ldots$ are pairwise disjoint.

We define the function $g_n$ on $[a_n, b_n]$ by

$$g_n(x) = A_n(x - a_n)^2 + B_n(x - a_n)^2 + (x - a_n)f'_E(a_n) + f(a_n).$$

where $A_n$ and $B_n$ are constants. We choose them such that $g_n(b_n) = f(b_n), g'_n(b_n) = f'_E(b_n)$. Then

$$A_n(b_n - a_n)^2 = \{f'_E(b_n) - f'_E(a_n)\} - 2 \left\{ \frac{f(b_n) - f(a_n)}{b_n - a_n} - f'_E(a_n) \right\},$$

$$B_n(b_n - a_n) = 3 \left\{ \frac{f(b_n) - f(a_n)}{b_n - a_n} - f'_E(a_n) \right\} - \{f'_E(b_n) - f'_E(a_n)\}.$$ 

We now define the functions $g$ and $h$ on the real line as follows:

$$g(x) = f(x), \quad h(x) = f'_E(x) \text{ for } x \in C,$$

$$g(x) = g_n(x), \quad h(x) = g'_n(x) \text{ for } a_n < x < b_n,$$

$$g(x) = (x - a)f'_E(a) + f(a), \quad h(x) = f'_E(a) \text{ for } x < a,$$

$$g(x) = (x - b)f'_E(b) + f(b), \quad h(x) = f'_E(b) \text{ for } x > b.$$
Clearly $g$ and $h$ are continuous at each point of the set $A = G \cup (-\infty, a) \cup (b, \infty)$; $g'(x)$ exists at each point of $A$ and $g'(x) = h(x)$. If $a \in C$, then

$$\lim_{\varepsilon \to 0} \frac{g(x) - g(a)}{x - a} = \lim_{\varepsilon \to 0} \frac{f(x) - f(a)}{x - a} = f'_E(a) = h(a),$$

This gives that $h$ is a derived function of $g$ on $(-\infty, \infty)$.

We now show that $g$ and $h$ are continuous at each point of $C$.

**Case 1.** Let $G = \cap_{n=1}^{\infty} (a_n, b_n)$.

Choose any $\eta > 0$. Since $f$ and $f'_E$ are uniformly continuous on $C$ and $f$ is uniformly differentiable on $C$, there is a $\delta$ with $0 < \delta < \min \{\eta, 1\}$ such that

$$\begin{cases} |f'(x) - f'(y)| < \eta, & |f'_E(x) - f'_E(y)| < \eta, \\ |f'(y) - f'(x)| < \eta, & \frac{|y - x|}{y - x} \to \frac{f'_E(x)}{f'_E(x)} < \eta, \end{cases}$$

(2.8)

for all $x, y \not= x$ in $C$ whenever $|x - y| < \delta$.

We choose a positive integer $N$ such that

$$\sum_{n=\mathbb{N}+1}^{\infty} (b_n - a_n) > \frac{1}{2} \delta. \tag{2.9}$$

Let $a \in C$. Suppose that $a \not\in \mathcal{B} = \{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}$ and let $\delta_a = \min \{\frac{1}{2} \delta, |a - u| : u \in \mathcal{B}\}$. Take any real number $x$ with $|x - a| < \delta_a$. If $x \in C$, then by (2.8)

$$|g(x) - g(a)| = |f(x) - f(a)| < \eta,$$

$$|h(x) - h(a)| = |f'_E(x) - f'_E(a)| < \eta.$$

If $x \in G$, then $x \in (a_n, b_n)$ for some $n > N$. We have, using (2.9), $b_n - a_n < \delta$ and $|a_n - a| < \delta$. So by (2.8)

$$|A_n(b_n - a_n)^2| \leq \left|f'_E(b_n) - f'_E(a_n)\right| + 2 \left|\frac{f(b_n) - f(a_n)}{b_n - a_n} - f'_E(a_n)\right| < 3\eta,$$

$$|B_n(b_n - a_n)| \leq 3 \left|\frac{f(b_n) - f(a_n)}{b_n - a_n} - f'_E(a_n)\right| + \left|f'_E(b_n) - f'_E(a_n)\right| < 4\eta,$$

$$|g(x) - g(a)| \leq |g(a_n) - g(a)| + |g(x) - f(a_n)|$$

$$\leq |f(a_n) - f(a)| + |A_n(b_n - a_n)^2| + |B_n(b_n - a_n)| + K(x - a_n)$$

$$< \eta + 3\eta + 4\eta + K\eta$$

$$< (8 + K) \eta.$$
where \( K = \text{Sup} \{ | f'_n(x) | : x \in C \} \).

\[
| h(x) - h(a) | \leq | h(a_n) - h(a) | + | h(x) - h(a_n) | \\
\quad \quad \leq | f'_n(a_n) - f'_n(a) | + 3 | A_n(b_n - a_n)^2 | + 2 | B_n(b_n - a_n) | \\
< \eta + 9\eta + 8\eta \\
= 18\eta.
\]

This gives that \( g \) and \( h \) are continuous at \( a \). Let \( a = a_r \) for some \( r (1 \leq r \leq N) \). Then clearly \( g \) and \( h \) are continuous at \( a \) on the right. Take

\[ \delta_0 = \min \left\{ \frac{1}{3} \delta, \frac{1}{2} | u - a | : u \in B \text{ and } u \neq a \right\} \]

Let \( x \) be any real number in \((a - \delta, a)\). If \( x \in C \), then \( | g(x) - g(a) | < \eta \) and \( | h(x) - h(a) | < \eta \). If \( x \in G \), then \( x \in (a_n, b_n) \) for some \( n > N \). As above we can show that

\[ | g(x) - g(a) | < (8 + k)\eta \text{ and } | h(x) - h(a) | < 18\eta. \]

Hence \( g \) and \( h \) are continuous at \( a \) on the left. If \( a = b_r \) for some \( r (1 \leq r \leq N) \), then as above we can show that \( g \) and \( h \) are continuous at \( a \).

**Case II.** Let \( G = \bigcup_{n=1}^{N} (a_n, b_n) \) for some positive integer \( N \). We may suppose that \( a < a_1 < b_1 < a_2 < b_2 < \ldots < a_N < b_N < b \). It is easy to see that \( g \) and \( h \) are continuous.

By Corollary 2.2.1, \( g'(x) \) exists for all \( x \in [a, b] \) and \( g'(x) = h(x) \). Thus \( g \) possesses continuous derivative on the whole real line and \( g'(x) = f'_n(x) \) for all \( x \in C \). This completes the proof of the theorem.

3. Results on approximate strong and approximate uniform differentiability

In this section we first show that a function approximately semi-continuous on a measurable set \( E \) is measurable; further if the function is finite a.e. on \( E \), then it is approximately continuous a.e. on \( E \).

**Definition 3.1:** An extended real-valued function \( f \) on the measurable set \( E \) is said to be approximately upper [lower] semi-continuous at \( a \in E \) if for every \( \epsilon > 0 \) there is a measurable set \( S_\varepsilon \subset E \) having \( a \) as point of density such that for all \( x \in S_\varepsilon \),

\[ f(x) \leq f(a) - \epsilon \cdot [f(x) > f(a) - \epsilon]. \]

If \( f \) is approximately continuous at \( a \), then it is obvious that \( f \) is approximately upper and lower semi-continuous at \( a \); conversely if \( f \) is approximately upper and lower semi-continuous at \( a \) and \( f(a) \neq \pm \infty \), then \( f \) is approximately continuous at \( a \).
Lemma 3.1: Let $A$ be any subset of the real line. If $A$ contains almost all of its density points, then $A$ is measurable.

Proof: Let $B$ denote the complement of $A$. Assume that $A$ is not measurable. Then the sets $A$ and $B$ are not metrically separated ([3], Ch. V, p. 117). So by Theorems 5.7 and 5.8 ([3], Ch. V, p. 117) there is a set $E \subset B$ such that $m^*E > 0$ and at each point of $E$ the density of $A$ is unity. This contradicts the hypothesis that $A$ contains almost all of its density points. Hence $A$ is measurable.

Theorem 3.1: (Cf. [2], p. 309). Let $f$ be an extended real-valued function on the measurable set $E$. If $f$ is approximately upper [lower] semi-continuous on $E$, then $f$ is measurable on $E$. Further if $f$ is finite a.e. on $E$, then $f$ is approximately continuous a.e. on $E$.

Proof: Suppose that $f$ is approximately upper semi-continuous on $E$. Let $\gamma$ be any real number and let $A = \{x : x \in E \text{ and } f(x) \geq \gamma\}$. Let $a$ be a point of density of $A$ and let $a \in E$. Assume that $f(a) < \gamma$. Since $f$ is approximately upper semi-continuous at $a$, there is a measurable set $S_a \subset E$ having $a$ as point of density such that

\[ f(x) < \gamma \text{ for all } x \in S_a. \tag{3.1} \]

Let $B = E - S_a$. From (3.1) we see that $A \subset B$. Clearly the density of $B$ at $a$ is zero which contradicts the fact that $A$ has unit density at $a$. Hence $f(a) \geq \gamma$, that is, $a \in A$. Since $A$ has density zero a.e. on $E'$ (complement of $E$) it follows that $A$ contains almost all of its density points. So by Lemma 3.1, the set $A$ is measurable. This gives that $f$ is measurable on $E$.

Further, if $f$ is finite a.e. on $E$, by Theorem 10.6 ([5], p. 132), $f$ is approximately continuous a.e. on $E$.

If $f$ is approximately lower semi-continuous on $E$, then $-f$ is approximately upper semi-continuous on $E$ and the result follows.

Definition 3.2: Let $f$ be a real-valued function on the set $E$. $f$ is said to be approximately strongly differentiable at $a \in E$, if there is a measurable set $S \subset E$ containing $a$ and having $a$ as point of density such that the double limit

\[ \lim_{(x,y) \to (a,a)} \frac{f(x) - f(y)}{|x - y|} \text{ exists finitely.} \]

The above limit whenever exists (finite or infinite) is denoted by $f_{s}^+ (a)$ and is called the approximate strong derivative of $f$ at $a$.

It is easy to see that if $f_{s}^+ (a)$ exists, then the approximate derivative $f_{a}^+ (a)$ exists and $f' (a) = f_{s}^+ (a)$. If $f_{s}^+ (a)$ does not exist, then $f_{a}^+ (a)$ may exist and $f' (a) = f_{a}^+ (a)$. If $f_{a}^+ (a)$ does not exist, then $f' (a)$ may exist and $f' (a) = f_{s}^+ (a)$. If $f' (a)$ does not exist, then $f_{s}^+ (a)$ and $f_{a}^+ (a)$ may exist and $f' (a) = f_{s}^+ (a)$. If $f' (a)$ and $f_{s}^+ (a)$ do not exist, then $f_{a}^+ (a)$ may exist and $f' (a) = f_{a}^+ (a)$. If $f_{a}^+ (a)$ and $f_{s}^+ (a)$ do not exist, then $f' (a)$ may exist and $f' (a) = f_{s}^+ (a)$.
Let $f$ be a real-valued function on the interval $I$ and let $f'\theta(x)$ exist finitely at each point of $I$. $f$ is said to be approximately uniformly differentiable at a point $\alpha \in I$ if for every $\varepsilon > 0$ there is a measurable set $S \subset I$ containing $\alpha$ and having $\alpha$ as point of density such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'\theta(x) \right| < \varepsilon$$

for all $x, y (x \neq y)$ in $S$.

For the remaining part of this section we suppose that $I$ is a fixed finite open interval, $f$ is a real-valued function on $I$ and $f'\theta(x)$ exists finitely at each point of $I$. This gives that $f$ is measurable on $I$ (see Theorem 3.1).

For each $\alpha \in I$ we denote by $F_\alpha$ the family of all measurable sets $S \subset I$ containing $\alpha$ and having $\alpha$ as point of density. For any two points $x, y (x \neq y)$ in $I$ we write

$$\psi(x, y) = \left| \frac{f(y) - f(x)}{y - x} - f'\theta(x) \right|.$$

For $\alpha \in I$ and $S \in F_\alpha$, let

$$\mathcal{U}(S, \alpha) = \sup \{\psi(x, y) : x, y \in S \text{ and } x \neq y\}.$$

Then clearly $\mathcal{U}(S_1, \alpha) \leq \mathcal{U}(S_2, \alpha)$ if $S_1 \subset S_2$.

Let $u(\alpha) = \inf \{\mathcal{U}(S, \alpha) : S \in F_\alpha\}$.

Theorem 3.2: The function $u$ is approximately upper semi-continuous on $I$ and hence $u$ is measurable on $I$.

Proof: Let $\alpha \in I$. Choose any $\varepsilon > 0$. There is an element $S_\alpha \in F_\alpha$ such that $\mathcal{U}(S_\alpha, \alpha) \leq u(\alpha) + \varepsilon$. Denote by $S$ the set of points of $S_\alpha$ where the density of $S_\alpha$ is unity. Then $S$ is measurable and each point of $S$ is a point of density of $S$. Clearly $S \in F_\alpha$ for each $\alpha \in S$. Take any $x \in S$. Since $S \supset S_\alpha$,

$$u(x) \leq \mathcal{U}(S, x) \leq \mathcal{U}(S, \alpha) \leq u(\alpha) + \varepsilon.$$

This gives that $u$ is approximately upper semi-continuous at $\alpha$ and so on $I$. By Theorem 3.1, $u$ is measurable on $I$.

Theorem 3.3: $f$ is approximately uniformly differentiable at $\alpha \in I$ iff $u(\alpha) = 0$.

Proof: First suppose that $f$ is approximately uniformly differentiable at $\alpha$. Choose any $\varepsilon > 0$. Then there is a member $S \in F_\alpha$ such that $\psi(x, y) < \varepsilon$ for all $x, y (x \neq y)$ in $S$. This gives that $\mathcal{U}(S, \alpha) \leq \varepsilon$. Since $u(\alpha) \leq \mathcal{U}(S, \alpha)$ we get $0 \leq u(\alpha) \leq \varepsilon$. It follows that $u(\alpha) = 0$. 

Next, let \( u(a) = 0 \). Choose any \( \varepsilon > 0 \). There is a member \( S \in F_\varepsilon \) such that \( U(S, a) < \varepsilon \). Since \( \psi(x, y) \leq U(S, a) \) for \( x, y \neq y \) in \( S \) we get
\[
\psi(x, y) < \varepsilon \text{ for all } x, y \text{ in } S \text{ with } x \neq y.
\]
Hence \( f \) is approximately uniformly differentiable at \( a \).

**Theorem 3.4:** For each \( k > 0 \), the measure of the set

\[ E = \{ x : x \in I \text{ and } u(x) \geq k \} \]

is zero.

**Proof:** Since \( u \) is measurable, the set \( E \) is measurable. Assume that \( mE > 0 \). Again, since \( f^{*}_{\alpha} (x) \) is finite at each point of \( E \), by Theorem 10.8 ([5], Ch. VII, p. 237) \( f \) is BV on \( E \). So there is a sequence of sets \( E_1, E_2, E_3, \ldots \) such that \( E = \bigcup_{n=1}^{\infty} E_n \) and \( f \) is BV on each \( E_n \). Since \( f \) is measurable on \( E \), from Theorem 4.2 ([5], Ch. VII, p. 222) it follows that the sets \( E_1, E_2, E_3, \ldots \) may be taken measurable. Since \( mE > 0, mE_\gamma > 0 \) for some positive integer \( \gamma \). Again, since \( f \) is BV on \( E_\gamma \), there is a function \( g \), BV on \( I \), such that \( g = f \) on \( E_\gamma \). Let \( A \) denote the set of points \( x \) of \( E_\gamma \) where \( f^{*}_{\alpha} (x) \) exists finitely and \( x \) is a point of density of \( E_\gamma \). Then \( A \) is dense in itself and \( mA = mE_\gamma > 0 \) and \( f^{*}_{\alpha} (x) \) exists finitely at each point of \( A \). By Theorem 2.3, there is a perfect set \( B \subset A \) such that \( nB > 0 \) and \( f \) is uniformly differentiable on \( B \). Choose any \( \varepsilon \) with \( 0 < \varepsilon < \varepsilon \)

then there is a \( \delta > 0 \) such that
\[
\left| \frac{f(y) - f(x)}{y - x} - f^{*}_{\alpha} (x) \right| < \varepsilon
\]

for all \( x, y (x \neq y) \) in \( B \) whenever \( |x - y| < \delta \).

Let \( C \) denote the set of points of \( B \) where the density of \( B \) is unity. Then \( mC = nB > 0 \). Let \( \alpha \in C \) and \( S = (\alpha - \delta, \alpha + \delta) \cap C \). Then \( S \in F_\alpha \) and \( f^{*}_{\alpha} (x) = f^{*}_{\alpha} (x) \)

or all \( x \in C \). We have from (3.2),
\[
\left| \frac{f(y) - f(x)}{y - x} - f^{*}_{\alpha} (x) \right| < \varepsilon
\]

for all \( x, y (x \neq y) \) in \( S \). This gives that \( U(S, \alpha) \leq \varepsilon \). So \( 0 \leq u(x) \leq \varepsilon \). This contradicts the hypothesis that \( u(x) \geq k \) for all \( x \in E \). Hence \( mE = 0 \).

**Theorem 3.5:** Let \( f \) be a real-valued function on the open interval \( I \) and let \( f^{*}_{\alpha} (x) \) exist finitely at each point of \( I \). Then \( f \) is approximately uniformly differentiable almost everywhere on \( I \).

**Proof:** For each positive integer \( n \), let
\[
E_n = \{ x : x \in I \text{ and } u(x) \geq 1/n \} \text{ and } E = \bigcup_{n=1}^{\infty} E_n. \] By Theorem 3.4, \( mE_n = 0 \)
for \( n = 1, 2, 3, \ldots \) and so \( mE = 0 \). If \( x \in I - E \), then \( u(x) = 0 \). So by Theorem 3.3 \( f \) is approximately uniformly differentiable at \( x \). This proves the theorem.

**Theorem 3.6**: \( f \) is approximately strongly differentiable at \( a \in I \) iff \( f \) is approximately uniformly differentiable at \( a \).

**Proof**: First suppose that \( f \) is approximately strongly differentiable at \( a \in I \). There is a member \( S_1 \in F_a \) such that
\[
\left| \frac{f(y) - f(x)}{y - x} - f'_a(a) \right| < \varepsilon \tag{3.3}
\]
for all \( x, y (x \neq y) \) in \( S_1 \).

Denote by \( S_2 \) the set of points of \( S_1 \) where the density of \( S_1 \) is unity. Then \( S_2 \) has unit density at each of its points. Let \( \omega \in S_2 \). Since \( f'_a(\omega) \) exists finitely, there is a member \( S_3 \in F_a \) such that
\[
\lim_{z \to \omega} \frac{f(z) - f(\omega)}{z - \omega} = f'_a(\omega).
\]
Take \( S_4 = S_2 \cap S_3 \). Then \( S_4 \in F_a \). Let \( z \in S_4 \) and \( z = \omega \). Then \( z, \omega \in S_1 \) and so by (3.3)
\[
\left| \frac{f(z) - f(\omega)}{z - \omega} - f'_a(a) \right| < \varepsilon.
\]
Letting \( z \to \omega \) over the set \( S_4 \) we have
\[
|f'_a(\omega) - f'_a(a)| \leq \varepsilon. \tag{3.4}
\]
Let \( x, y \in S_3 \) and \( x \neq y \). Then
\[
\left| \frac{f(y) - f(x)}{y - x} - f'_a(x) \right| \leq \left| \frac{f(y) - f(x)}{y - x} - f'_a(a) \right| + |f'_a(x) - f'_a(a)|
\]
\[
< 2\varepsilon \text{ [using (3.3) and (3.4)]}.
\]
This gives that \( 0 \leq u(a) \leq U(S_2, a) \leq 2\varepsilon \). Since \( \varepsilon > 0 \) is arbitrary we obtain \( u(a) = 0 \). Hence \( f \) is approximately uniformly differentiable at \( a \).

Next, let \( f \) be approximately uniformly differentiable at \( a \). Choose any \( \varepsilon > 0 \). There is a member \( S \in F_a \) such that
\[
\left| \frac{f(y) - f(x)}{y - x} - f'_a(x) \right| < \varepsilon/3 \tag{3.5}
\]
for all \( x, y (x \neq y) \) in \( S \).
Let $x \in S$ and $x \neq a$. Then by (3.5),

$$|f_{ap}^{*}(x) - f_{ap}^{*}(a)|$$

$$\leq \left| \frac{f(x) - f(a)}{x - a} - f_{ap}^{*}(a) \right| + \left| \frac{f(a) - f(x)}{a - x} - f_{ap}^{*}(x) \right| < \frac{2}{3} \varepsilon. \quad (3.6)$$

Again, for $x, y (x \neq y)$ in $S$,

$$\left| \frac{f(y) - f(x)}{y - x} - f_{ap}^{*}(a) \right| \leq \left| \frac{f(y) - f(x)}{y - x} - f_{ap}^{*}(x) \right| + \left| f_{ap}^{*}(x) - f_{ap}^{*}(a) \right|$$

$$< \frac{1}{3} \varepsilon + \frac{2}{3} \varepsilon = \varepsilon \quad [\text{using (3.5) and (3.6)}].$$

This gives that $f$ is approximately strongly differentiable at $a$.

From Theorems 3.5 and 3.6 we obtain the following:

**Theorem 3.7**: Let $f$ be a real-valued function on the finite open interval $I$ and let $f_{ap}^{*}(x)$ exist at each point of $I$. Then $f$ is approximately strongly differentiable almost everywhere on $I$.

### 4. Essentially BV and AC functions

**Definition 4.1**: Let $f$ be a real-valued function on the measurable set $E$. $f$ is said to be essentially $BV\ [AC]$ on $E$ if given any $\varepsilon > 0$ there is a measurable set $A \subset E$ with $m(E - A) < \varepsilon$ such that $f$ is $BV\ [AC]$ on $A$.

Let $E$ be a measurable set with $mE < +\infty$ and let $f$ be essentially $AC$ on $E$. Then it is easy to show that $f$ is essentially $BV$ on $E$. In this section we show that the converse is also true. Further we show that a measurable function $BVG$ on a set $E$ with $mE < +\infty$ is essentially $AC$ on $E$.

Throughout this section we suppose that $E$ is a measurable set with $mE < +\infty$ and $f$ is a real-valued function measurable on $E$.

**Theorem 4.1**: If $f$ is approximately strongly differentiable a.e. on $E$, then $f$ is essentially $AC$ on $E$.

**Proof**: Denote by the set of points of $E$ where $f$ is approximately strongly differentiable. Then $m(E - B) = 0$.

Let $\varepsilon > 0$ be chosen arbitrarily. Take $\eta = \varepsilon/(1 + mE)$. Let $a \in B$. Then there is a measurable set $S_{\varepsilon} \subset B$ containing $a$ and having $a$ as point of density such that

$$\left| \frac{f(y) - f(x)}{y - x} - f_{ap}^{*}(a) \right| < \varepsilon$$
for all $x, y \ (x \neq y)$ in $S_a$. Then

$$|f(y) - f(x)| \leq \{1 + |f_{x_0}(a)|\} |y - x|$$

for all $x, y$ in $S_a$. This gives that $f$ is $AC$ on $S_a$. Since $a$ is a point of density of $S_a$, there is a positive number $\delta_a$ such that

$$\frac{m[S_a \cap \Delta_a(\delta)]}{m\Delta_a(\delta)} > 1 - \eta$$

for all $\delta$ with $0 < \delta \leq \delta_a$, where $\Delta_a(\delta) = [a - \delta, a + \delta]$.

Let $F = \{\subseteq_a(\delta) : 0 < \delta \leq \delta_a$ and $a \in B\}$. Then $F$ covers the set $B$ in the sense of Vitali. Hence by Vitali's Theorem ([3], Ch. V, Th. 5.1, p. 110), there is a finite number of pairwise disjoint intervals

$$\Delta_a_1(h_1), \Delta_a_2(h_2), \ldots, \Delta_a_N(h_N) \ (a \in B)$$

in the family $F$ such that

$$\sum_{i=1}^{N} m[B \cap \Delta_a_i(h_i)] > mB - \eta. \quad (4.2)$$

Write $A_i = S_a \cap \subseteq_a_i(h_i) \ (i = 1, 2, \ldots, N)$ and $A = \bigcup_{i=1}^{N} A_i$.

By (4.1) we have

$$mA_i > (1 - \eta) m[\Delta_a_i(h_i)] \Rightarrow (1 - \eta) m[B \cap \Delta_a_i(h_i)]$$

and

$$mA = \sum_{i=1}^{N} mA_i > (1 - \eta) \sum_{i=1}^{N} m[B \cap \Delta_a_i(h_i)]$$

$$> (1 - \eta) (mB - \eta) \quad [\text{using (4.2)}]$$

$$> mB - \eta (1 + mB) = mB - \varepsilon = mE - \varepsilon.$$

The function $f$ is $AC$ on each of the sets $A_1, A_2, \ldots, A_N$. Since the intervals $\Delta_a_1(h_1), \Delta_a_2(h_2), \ldots, \Delta_a_N(h_N)$ are pairwise disjoint we can show that $f$ is $AC$ on $A$. Hence $f$ is essentially $AC$ on $E$.

**Theorem 4.2**: If $f$ is BVG on $E$, then $f$ is essentially $AC$ on $E$; hence essentially BV on $E$.

**Proof**: Denote by $B$ the set of points of $E$ where $f$ is approximately strongly differentiable. Since $f$ is BVG and measurable on $E$, $E$ can be expressed in the form $E = \bigcup_{s=1}^{\infty} E_s$, where $f$ is BV on each $E_s$ and each $E_s$ is measurable. For each $n, f_{x_0}(a)$ exists finitely a.e. on $E_s$. Choose any $\varepsilon > 0$. By Theorem 2.3 there is a perfect set $C_n \subset E_s$ with
$m(E_n - C_n) < \varepsilon/2^n$ such that $f$ is uniformly differentiable on $C_n$. Denote by $A_n$ the set of points of $C_n$ where the density of $C_n$ is unity. Then each point of $A_n$ is a point of density of $A_n$ and $m(E_n - A_n) < \varepsilon/2^n$. Let $A = \bigcup_{n=1}^{\infty} A_n$. Then $A \subset E$ and $m(E - A) < \varepsilon$.

Let $a \in A$. Then $a \in A_n$ for some $n$. Choose any $\eta > 0$. Since $f$ is uniformly differentiable on $C_n$ there is a $\delta > 0$ such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'_a(x) \right| < \frac{1}{3} \eta \quad (4.3)$$

for all $x, y$ $(x \neq y)$ in $C_n$. Take $S = (a - \delta, a + \delta) \cap C_n$. Then $S$ is measurable, $a \in S$ and $S$ has unit density at $a$. If $x \in S$ and $x \neq a$, then using (4.3) we get

$$\left| f'_{E_n}(x) - f'_{E_n}(a) \right| \leq \left| \frac{f(x) - f(a)}{x - a} - f'_a(a) \right| + \left| \frac{f(a) - f(x)}{a - x} - f'_E(x) \right| < \frac{2}{3} \eta. \quad (4.4)$$

Therefore for any $x, y$ $(x \neq y)$ in $S$,

$$\left| \frac{f(y) - f(x)}{y - x} - f'_E(a) \right| \leq \left| \frac{f(y) - f(x)}{y - x} - f'_E(x) \right| + \left| f'_E(x) - f'_E(a) \right|$$

$$< \eta \text{ [using (4.3) and (4.4)].}$$

This gives that $f$ is approximately strongly differentiable at $a$ and so $a \in B$. Hence $A \subset B$. We have $E - B \subset E - A$. So $m(E - B) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $m(E - B) = 0$. Thus $f$ is approximately strongly differentiable a.e. on $E$. By Theorem 4.1, $f$ is essentially $AC$ on $E$.

**Corollary 4.2.1:** If $f$ is essentially $BV$ on $E$, then $f$ is essentially $AC$ on $E$.

**Proof:** For each positive integer $n$, there is a measurable set $B_n \subset E$ with $m(E - B_n) < 1/n$ such that $f$ is $BV$ on $B_n$. Let $B = \bigcup_{n=1}^{\infty} B_n$. Then $B \subset E$ and $m(E - B) = 0$. The function $f$ is $BVG$ on $B$. So by Theorem 4.2, $f$ is essentially $AC$ on $B$ which gives that $f$ is essentially $AC$ on $E$.

**Corollary 4.2.2:** If $f'_a(x)$ exists finitely a.e. on $E$, then $f$ is essentially $AC$ on $E$.

**Proof:** Let $B$ denote the set of points of $E$ where $f'_a(x)$ exists finitely. Then $m(E - B) = 0$. By Theorem 10.8 ([5], Ch. VII, p. 237) $f$ is $BVG$ on $B$. Now by Theorem 4.2, $f$ is essentially $AC$ on $B$ and so on $E$.

**Theorem 4.3.** If $f'_a(x)$ exists finitely a.e. on $E$, then given any $\varepsilon > 0$ there is a perfect set $A \subset E$ with $m(E - A) < \varepsilon$ such that $f'_a(x)$ exists finitely at each point of $A$ and that $f'$ is uniformly differentiable on $A$. 
Proof: By Corollary 4.2.2, $f$ is essentially $BV$ on $E$. Choose any $\varepsilon > 0$. There is a measurable set $B \subset E$ with $m(E - B) < \frac{1}{2} \varepsilon$ such that $f$ is $BV$ on $B$. Denote by $C$ the set of the points of $B$ where $f_B'(x)$ exists finitely. Then $m(B - C) = 0$. By Theorem 2.3, there is a perfect set $A \subset C$ with $m(C - A) < \frac{1}{2} \varepsilon$ such that $f$ is uniformly differentiable on $A$. We have $A \subset C \subset B \subset E$. So $E - A = (E - B) \cup (B - C) \cup (C - A)$ and $m(E - A) \leq m(E - B) + M(B - C) + m(C - A) < \varepsilon$.

This proves the theorem.

From Theorems 4.3 and 2.4 we obtain the following:

**Theorem 4.4:** If $f_B'(x)$ exists finitely a.e. on $E$, then given any $\varepsilon > 0$, there is a perfect set $A \subset E$ with $m(E - A) < \varepsilon$ and a function $g$ defined on the real line having the following properties.

(i) $g$ possesses continuous derivative on the real line.

(ii) $g(x) = f(x)$ and $g'(x) = f_B'(x)$ for all $x \in A$.

5. **Notations**

a. e. = almost everywhere

iff = if and only if

**References**


