Nonnegative solution of linear equations

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Abstract

We give a method to obtain a nonnegative solution of any system of linear equations, if such a solution exists. The method writes linear equations as a linear programming problem and then solves this problem using a Simplex method.

Key words: Artificial basis technique, linear programming, nonnegative solution, Simplex method.

1. Introduction

In many physical problems, the negative of quantities like path, matter, time, etc., does not arise. Any such problem giving rise to linear equations involving such unknown quantities needs nonnegative solution.

The method described here investigates equations \( By = g \), consistent or not, underdetermined or overdetermined, as a linear programming (lp) problem and gives a nonnegative solution \( y \) when it exists. To solve the \( lp \) problem the method involves a particular form of the artificial basis technique\(^2\).

2. Definitions

Extended (Simplex) tableau

Consider the \( lp \) problem

\[
\text{Minimize } f = c'x \text{ subject to } Ax = b, \ x \geq 0.
\] (1)

The initial extended tableau (i.e., Extended tableau 0) for this \( lp \) problem, where

\[
d_j = c_{n+1}d_{1j} + c_{n+2}d_{2j} + \cdots + c_{n+m}d_{mj} - c_j, \ j = 1 (1) n
\]

\[
d_{n+1} = c_{n+1}b_{1j} + c_{n+2}b_{2j} + \cdots + c_{n+m}b_{mj}.
\]
The function $J = f(s) = -c's$, or equivalently, $f(x) = c, x, 1 - Ic,.,$, which is to be optimized (minimized or maximized) is called the objective function.

The foregoing relationship between $d, a, c,,$ holds in all tableaux. This relationship is referred to as the Checking rule for a tableau. Satisfaction of this rule is necessary for a tableau to be correct but it is not sufficient (i.e., the rule may be satisfied even if a computational mistake occurs).

Note: The role of $c_1, ..., c_n, m$ is over as soon as $d_1, ..., d_{n+m}$ are computed. The subsequent extended tableaux (viz., Extended tableaux 1, 2, ..) are computed from the Extended tableau 0 using Simplex rules. The $d_1$-row has to be nonpositive (for the minimization problem considered here) in the optimal (final) Extended tableau.

Restricted (Simplex) tableau

Consider the $lp$ problem (1). The initial restricted tableau (i.e., Restricted tableau 0) for this $lp$ problem is the Extended tableau 0 with columns containing unit vectors deleted. The subsequent restricted tableaux are computed from the Restricted tableau 0 using Simplex rules? The $d_1$-row has to be nonpositive (for the minimization problem considered here) in the optimal (final) Restricted tableau.

3. The problem

Obtain a nonnegative solution of $By = g$ (if it exists)
where $B = (b_{ij})$ is a given $m \times n$ matrix, $g = (g_i)$ is a given nonnegative $m$-vector, and $y = (y_i)$ is an $n$-vector.
Note: There is no loss of generality in considering \( g > 0 \). If this is not so, then multiply the equations with negative \( g \), by \(-1\).

4. Existence of a nonnegative solution

\( By = g \) has a nonnegative solution \( y \) if and only if \( B^t z \geq 0 \), \( g^t z < 0 \) has no solution \( z \). Equivalently, \( By = g \) has no nonnegative solution \( y \) if and only if \( B^t z \geq 0 \), \( g^t z < 0 \) has a solution \( z \).

For proof see Farkas and Vajda.

This result is not of immediate use. However, the method tells if a nonnegative solution of \( By = g \) does not exist. In fact, the necessary and sufficient condition for \( By = g \) to have a nonnegative solution is the method producing one.

5. The method

Write (3) as an \( Ip \) problem and solve this problem using an artificial basis technique, as below:

(i) Equivalent \( Ip \) problem

Let \( y \) and \( B \) be now extended to \( (n + m) \)-vector \( x \) and \( m \times (n + m) \) matrix \( A \), respectively. Further, let the last \( m \) columns of \( A \) form an \( m \times m \) unit matrix \( I_m \). The \( Ip \) problem equivalent to (3) is

\[
\begin{align*}
\text{Min} \ f &= x_{n+1} + \cdots + x_{n+m} = 0: \text{Objective function} \\
\text{subject to} \\
Ax - b &= 0: \text{Constraints} \\
x &\geq 0: \text{Nonnegativity conditions}
\end{align*}
\]

(ii) Artificial basis technique in 'Extended tableau'

Step 1: Set up the extended Simplex tableau for (4), and write the coefficients (in parentheses) which \( x_i \) have in the objective function and the last row, i.e., \( d_j \)-row using the Checking rule as below:

\[
\begin{pmatrix}
(0) & (0) & (0) & (1) & (1) & (1) \\
x_1 & \cdots & x_{n+1} & x_{n+1} & \cdots & x_{n+m} \\
(1) x_{n+1} & a_{11} & a_{12} & a_{1n} & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
\]
Step 2 (pivot selection): Let \( d_{ij} \) be positive. Consider then, for all positive \( a_{ij} \), the ratios \( b_i/a_{ij} \) and take the smallest. If this is obtained for \( i_o \) then call \( p = a_{i_o j_o} \) the pivot (marked with a plus). Go to Step 3. Otherwise (i.e., if there exists no \( d_{ij} \) which is positive) the present tableau is final and it either indicates no solution of \( By = g \) or gives a solution.

Step 3 (next-tableau computation): Replacing \( x_{n+1} \) by \( x_{i_o} \) obtain the next tableau as follows:

\[
\begin{align*}
\begin{array}{ccccccccccc}
& x_1 & \cdot & \cdots & x_n & x_{n+1} & \cdots & x_{n+i} & \cdots & x_{n+m} & b \\
\end{array} \\
\begin{array}{ccccccccccc}
x_{n+1} & 0 = a_{11} & - p \cdot a_{1k}/p & - a_{1j} \\
\vdots & \vdots & & \vdots \\
x_{i_o} & a_{i_o j}/p & 1 & a_{i_o k}/p & 0 & 1/p & 0 & b_{i_o}/p \\
\vdots & \vdots & & \vdots \\
x_{n+m} & 0 = a_{m1} & - p \cdot a_{mk}/p & - a_{mj}/p & - d_j/p \\
\end{array}
\end{align*}
\]

The blank positions are filled in as follows:

\[
\begin{align*}
a_{ij} & \leftarrow a_{ij} - a_{i o j} a_{ij}/p \\
d_j & \leftarrow d_j - d_{i o j} a_{ij}/p \\
b_{i_o} & \leftarrow b_{i_o} - a_{i o j} b_{i o}/p.
\end{align*}
\]

All the entries on the right hand side of (7) are the elements of the previous tableau.

Both (6) and (7) may be precisely written as \( p = a_{i_o j_o} \)

\[
\begin{align*}
pivot row & \leftarrow pivot row/p \\
\text{(any other) } i\text{-th row} & \leftarrow i\text{-th row} \times pivot row
\end{align*}
\]
Note: Pivot row is the row containing the pivot. Pivot column is the column containing the pivot.

Step 4 (termination condition): If the bottom row, i.e., \( d_r \)-row excluding the last element is nonpositive, and if none of \( x_{n+1}, \ldots, x_{n+m} \) occurs in the basis with a nonzero value then the solution is reached. Otherwise go to Step 2.

(iiia) Artificial basis technique in 'restricted tableau'

Step 1 a: Set up the restricted Simplex tableau for (4), and write the coefficients (in parentheses) which \( x_j \) have in the objective function and the last row, i.e., \( d_r \)-row using the Checking rule as below.

\[
\begin{array}{cccc}
0 & 0 & 0 & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \vdots \\
\end{array}
\]

Note: Here corresponds to (I) of the extended Simplex tableau (2).

Step 2 a (pivot selection): It is the same as Step 2 with replacement of tableau (5) by tableau (5 a).

Step 3 a (next-tableau computation): Having interchanged \( x_{i_0} \) and \( x_{n+i_0} \) obtain the next tableau as follows:

\[
\begin{array}{cccc}
0 & 0 & 0 & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \vdots \\
\end{array}
\]
The blank positions are filled in as follows:

\[ a_{ij} \leftarrow a_{ij} - a_{ij}^d a_{ij}^d / p \]
\[ d_i \leftarrow d_i - a_{ij}^d d_i^d / p \]

(7 a)

**Note:**

- The foregoing two 'replacements' are actually identical when we consider the last row (i.e., \( d_j \)-row) as just another row like the rows of \((a_{ij})\).
- The right hand side elements are the elements of the foregoing tableau throughout the computation.

**Step 4 a (termination condition):** It is the same as Step 4.

6. **Proof of the method**

The method is a particular case of the M-method\(^4\). So, all the properties and inferences regarding the M-method hold good here also. However, all the tableaux in solving the problem are equivalent in the sense that the solution or solutions of the original equations remain invariant throughout. If there exists a nonnegative solution then the final tableau will give it. On the contrary, if there is none then one or more artificial variables will be in the basis (in the final tableau) with a nonzero value.

7. **Examples**

(i) **Nondegenerate case** (i.e., rank of coefficient matrix = number of equations = 2): Obtain a nonnegative solution of

\[
\begin{bmatrix}
1 & 2 & 1 & 1 & 0 \\
-4 & -2 & 3 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
=\begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

The equivalent LP problem is: Compute \( x = (x_1, x_2, x_3, x_4, x_5, x_7) \) so that \( \text{Min } f = x_4 + x_7 = 0 \) subject to \( Ax = b, \ x \geq 0 \) where

\[
A = \begin{bmatrix}
1 & 2 & 1 & 1 & 0 & 1 & 0 \\
-4 & -2 & 3 & 0 & 1 & 0 \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

We now write both extended and restricted tableaux to obtain the solution and to show how the restricted tableau differs from the extended one.
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### Restricted tableau 0

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Hence a nonnegative solution is $x = [\frac{2}{3}, 0, 0, 0]^t$.

(ii) **Degenerate case** (redundant equations): Obtain a nonnegative solution of

\[
\begin{bmatrix}
-1 & 2 & 3 & 3 \\
2 & 5 & 6 & 3 \\
-5 & -8 & -9 & -3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= 
\begin{bmatrix}
7 \\
16 \\
-25
\end{bmatrix}
\]

Setting up the equivalent LP problem the restricted tableaux are

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### Extended tableau

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<td>$-\frac{3}{2}$</td>
<td>$\frac{3}{2}$</td>
<td>$1$</td>
<td>$\frac{1}{2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_4$</td>
<td>4</td>
<td>$1^+$</td>
<td>$-2$</td>
<td>$-3$</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_7$</td>
<td>8</td>
<td>2</td>
<td>$-3$</td>
<td>$-6$</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Note: The last equation has been multiplied by $-1$ to make $b_2$ positive (refer Restricted tableau 0).

**Restricted tableau 2**

\[
\begin{array}{ccccc}
  x_1 & x_2 & x_3 & x_4 & b \\
  x_3 & -3 & -\frac{3}{8} & \frac{3}{8} & 3 \\
  x_4 & 4 & 1 & -2 & -3 \\
  x_5 & 0 & -2 & 1 & 0 \\
  -12 & -3 & 0 & 0 & 0 \\
\end{array}
\]

The artificial variable $x_7$ remains in the basis with a zero value. A nonnegative solution is $x = [0 \ 2 \ 1 \ 0]^T$.

(iii) **Inconsistent equations**: Obtain a nonnegative solution (if any) of

\[
\begin{bmatrix}
5 & 3 & 2 \\
2 & 1 & 2 \\
4 & 2 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
10 \\
5 \\
1
\end{bmatrix}
\]

Setting up the equivalent LP problem we write the restricted tableaux as below:

**Restricted tableau 0**

\[
\begin{array}{ccccc}
  x_1 & x_2 & x_3 & b \\
  x_4 & 1 & -\frac{3}{8} & -4 & \frac{1}{2} \\
  x_5 & 0 & 1 & 0 & \frac{9}{2} \\
  x_6 & 2 & \frac{1}{2} & 2 & \frac{1}{2} \\
  11 & 6 & 8 & 16 & \\
\end{array}
\]

The last row except the last element (viz., 13) is nonpositive and two artificial variables, viz., $x_4$ and $x_5$ are still in the basis with nonzero values. Hence the equations have no nonnegative solution. In fact, the equations have no solution at all.

(iv) **Solution with one negative element**: Obtain a nonnegative solution (if any) of

\[
\begin{bmatrix}
2 & 1 & 3 \\
1 & 1 & 1 \\
1 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
1 \\
2
\end{bmatrix}
\]
Nonnegative solution of linear equations

Restricted tableau 0

\[
\begin{array}{cccccc}
(0) & (0) & (0) \\
\hline
x_1 & x_2 & x_3 & b \\
(1) x_4 & 2 & 1 & 3^+ & 2 \\
(1) x_5 & 1 & 1 & 1 & 1 \\
(1) x_6 & 1 & 2 & 1 & 2 \\
& 4 & 4 & 5 & 5 \\
\end{array}
\]

Restricted tableau 1

\[
\begin{array}{cccccc}
(0) & (0) & (0) \\
\hline
x_1 & x_2 & x_3 & b \\
(1) x_4 & 2 & 1 & 3^+ & 2 \\
(1) x_5 & 1 & 1 & 1 & 1 \\
(1) x_6 & 1 & 2 & 1 & 2 \\
& 4 & 4 & 5 & 5 \\
\end{array}
\]

Restricted tableau 2

\[
\begin{array}{cccccc}
x_3 & x_2 & x_4 & b \\
-2 & -\frac{1}{2} & 1 & 0 \\
x_1 & 3 & 2^+ & -1 & 1 \\
x_6 & -1 & 1 & 0 & 1 \\
-2 & 1 & -1 & 1 \\
\end{array}
\]

Restricted tableau 3

\[
\begin{array}{cccccc}
x_3 & x_2 & x_4 & b \\
-\frac{3}{2} & -\frac{1}{2} & 1 & 0 \\
x_2 & 3^+ & -\frac{1}{2} & 1 & 1 \\
x_6 & -\frac{3}{2} & -\frac{1}{2} & 1 & 1 \\
-\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 & 1 \\
\end{array}
\]

The last row except the last element (viz., 1/2) is nonpositive and one artificial variable, viz., \( x_6 \) is still in the basis with nonzero value 1/2. Hence the equations have no nonnegative solution. However, the solution is \( x = [-1 \ 1 \ 1]^{T} \).

8. Acknowledgement

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Nomenclature

Symbol | Meaning
---|---
\( ^{\leftarrow} \) | is replaced by
\( B = (b_{ij}) \) | \( m \times n \) matrix
\( y = (y_{j}) \) | \( n \)-vector
\( g = (g_{i}) \) | \( m \)-vector
\( I_{n} \) | unit matrix of order \( m \)
\( A = (a_{ij}) \) | \( m \times (n + m) \) matrix, \( A = (B, I_{m}) \)
\( t \) | transpose
\[ x = (x_j) \quad (n + m)\text{-vector}, \quad x = [y_1 \cdots y_n x_{n+1} \cdots x_{n+m}]^T \]

\[ z = (z_j) \quad m\text{-vector} \]

\[ b = (b_k) \quad m\text{-vector}, \quad b = g \]

\[ c = (c_j) \quad (n + m)\text{-vector} \]

\[ d_j \quad j\text{-th element of the } d_j\text{-row}, \]

\[ d_j = c_{k+1}a_{ij} + c_{k+2}a_{2j} + \cdots + c_{n+m}a_{mj} - c_j \]

\( lp \quad \text{linear programming} \)

References


