Short Communication

2 X 2 Matrix-multiplication revisited

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Abstract

In this note, an algorithm for 2 X 2 matrix-multiplication is described, and an application of this is made to 3 X 3 matrix-multiplication.

Key words: Matrix-multiplication, computer algorithm.

1. Introduction

Strassen's algorithm\(^1\) for multiplying two 2 X 2 matrices, with entries from an arbitrary ring \(R\), involves 7 multiplications and 18 additions (assuming that addition and subtraction are the same kind of operations). Subsequently, Winograd\(^2\) discovered a more efficient algorithm, involving 7 multiplications but only 15 additions. In this note, we give an alternative algorithm, which also involves 7 multiplications and 15 additions, and a combination of the two algorithms is applied to 3 X 3 matrix-multiplication.

2. Algorithm

Let

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}
\]

Now with the Electrical Engineering Department.
be matrices, where \( a_{ij}, b_{ij} \in \mathbb{R}, 1 \leq i, j \leq 2 \). Then, the alternative algorithm is given by the identities below:

\[
\begin{align*}
    a_{11}b_{11} + a_{12}b_{21} &= t + (a_{12} - a_{22})(b_{21} - b_{22}) + (a_{12} - a_{11})(b_{21} - b_{22}) \\
    a_{11}b_{12} + a_{12}b_{22} &= t + a_{11}[(b_{12} - b_{11}) - (b_{21} - b_{22})] + (a_{12} - a_{22})(b_{11} + b_{21}) \\
    a_{21}b_{11} + a_{22}b_{21} &= t + b_{11}[(a_{21} - a_{11}) + (a_{12} - a_{22})] + (a_{12} - a_{11})(b_{21} - b_{22})
\end{align*}
\]

where \( t = a_{22}b_{22} + (a_{11} - (a_{12} - a_{22}))(b_{11} + (b_{21} - b_{22})) \). The term \( a_{21}b_{12} + a_{22}b_{22} \) is computed as it is. If intermediate results are appropriately saved, it is easy to see that the algorithm requires 7 multiplications and 15 additions.

3. Application

To compute the product of the matrices, \( P = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} \) and \( Q = \begin{bmatrix} q_{11} \\ q_{21} \end{bmatrix} \), \( p_{ij}, q_{ij} \in \mathbb{R}, 1 \leq i, j \leq 3 \), we have to compute the terms \( \sum_{j=1}^{3} p_{ij}q_{ik}, 1 \leq i, k \leq 3 \). This can be done in \( 5 \) multiplications, by combining Winograd’s scheme with ours.

We first note that in Winograd’s algorithm the term \( a_{11}b_{11} + a_{12}b_{21} \) of the product \( AB \) is computed as it is.

The partial sums \( \sum_{j=1}^{2} p_{ij}q_{jk}, 1 \leq i, k \leq 2 \), can be computed by multiplying the matrices

\[
\begin{bmatrix}
    p_{11} & p_{12} \\
    p_{21} & p_{22}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
    q_{11} & q_{12} \\
    q_{21} & q_{22}
\end{bmatrix},
\]

according to the above algorithm in 7 multiplications. The partial sums \( \sum_{j=2}^{3} p_{ij}q_{jk}, 2 \leq i, k \leq 3 \), can be computed by multiplying the matrices

\[
\begin{bmatrix}
    p_{12} & p_{13} \\
    p_{22} & p_{23}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
    q_{22} & q_{23} \\
    q_{32} & q_{33}
\end{bmatrix}
\]

by Winograd’s algorithm in 6 more multiplications, since \( p_{22}q_{22} \) is available from the first step; 12 more multiplications are needed to compute all the terms of \( PQ \), and this brings the tally to 25. This result was found by Gastinel in a more involved way.

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References

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