Bivariate Laplace transforms for some $H$-functions

B. L. MATHUR
Defence Laboratory, Ratanada Palace, Jodhpur 342 001, India

Received on December 26, 1977

Abstract

The object of the present paper is to have computation of certain bivariate Laplace transforms for the $H$-functions. A general theorem is established, which gives the correspondences, involving Fox's $H$-function, between the original and the image in two variables. A few particular cases of interest are also discussed.

Key words : Computation, Laplace transform, Fox's $H$-function, image and original functions, operational correspondences, integral equation.

1. Introduction

The Laplace Carson transform in two variables is defined and represented by the integral equation (p. 39)

$$F(p, q) \equiv pq \int_0^\infty \int_0^\infty e^{-ps-qt} f(x, y) \, dx \, dy; \quad \text{Re} (p, q) > 0;$$

where $F(p, q)$ and $f(x, y)$ are said to be operationally related to each other. $F(p, q)$ is called the image and $f(x, y)$ the original.

Symbolically we can write

$$F(p, q) \equiv f(x, y) \quad \text{or vice versa,}$$

where the symbol $\equiv$ is termed as operational.

The $H$-function is defined and represented in the notation of Braaksma as follows:

$$H^m_{r, s} \left[ x \mid (a, A) \right]_n = H^m_{r, s} \left[ x \mid (a_1, A_1), \ldots, (a_r, A_r) \right]_n$$

$$= (2i\pi)^{-1} \int_L \frac{\prod_{j=1}^{r} \Gamma(b_j - wB_j) \prod_{j=1}^{n} \Gamma(1 - a_j + wA_j)}{\prod_{j=n+1}^{s} \Gamma(1 - b_j + wB_j) \prod_{j=1}^{r} \Gamma(a_j - wA_j)} x^w \, dw,$$
where $x$ may be real or complex but is not equal to zero and an empty product is interpreted as unity, $m, n, r, s$ are integers satisfying the inequalities $1 \leq m \leq r, 0 \leq n \leq r, A_j (j = 1, \ldots, r), B_j (j = 1, \ldots, s)$ are positive numbers and $A_i (j = 1, \ldots, r)$, $b_j (j = 1, \ldots, s)$ are complex numbers such that no pole of $\Gamma (b_k - w A_j) (k = 1, \ldots, n)$ coincides with any pole of $\Gamma (1 - a_i + w A_j) (j = 1, \ldots, n)$, i.e.,

$$A_j (b_k + \epsilon) \neq B_k (a_i - \eta - 1)$$

$(\epsilon, \eta = 0, 1, 2, \ldots; h = 1, \ldots, m; j = 1, \ldots, n)$.

The contour $L$ runs from $\sigma - i \infty$ to $\sigma + i \infty$ ($\sigma$ real) such that the poles of $\Gamma (b_i - w A_j)$ ($j = 1, \ldots, n$) lie on the right-hand side of $L$ and those of $\Gamma (1 - a_i + w A_j)$ ($j = 1, \ldots, n$) lie on the left-hand side of $L$. Such a contour is possible on account of (4).

The integral in (3) converges for

$$| \arg x | < \frac{1}{2} \pi, \lambda > 0,$$

where

$$\lambda = \sum_{j=1}^{r} A_j - \sum_{j=r+1}^{s} A_j + \sum_{j=1}^{r} B_j - \sum_{j=r+1}^{s} B_j.$$

These conditions are assumed to hold good throughout this paper.

In this paper we shall obtain correspondences, involving Fox's $H$-function, between the original and the image in two variables.

In what follows we shall denote the original variables by $x$ and $y$ and the transformed variables by $p$ and $q$. The notations employed are those of Ditkina and Prudnikov's Operational Calculus. The results obtained here provide a generalization to the results given earlier by Dahiya.

2. Theorem

If

(i) $0 \leq n \leq r, 1 \leq m \leq s, \delta > 0, \Re (p) > 0$,

(ii) $| \arg u | < \frac{1}{2} \pi \lambda, \lambda > 0$,

(iii) $\Re (\nu) > 0, \Re [\beta + \delta b_j / B_j] > -1 (j = 1, \ldots, m),\]

$$\Re [\beta + \nu - \delta (a_j - 1) / A_j] < 3/4 (j = 1, \ldots, n),$$

(iv) $A_j (b_k + \epsilon) \neq B_k (a_i - \eta - 1)$

$(\epsilon, \eta = 0, 1, 2, \ldots; h = 1, \ldots, m; j = 1, \ldots, n)$,
BIVARIATE LAPLACE TRANSFORMS FOR SOME $H$-FUNCTIONS

\[ p^{-\frac{1}{2}} (pq)^{\frac{(3/2) - 1}{2}} \beta^{-v} H_{r+1, s+1}^{m, n+1} \left[ (u^2 pq)^{\frac{1}{2}} \right] \left[ (\beta, \delta), (a, A) \right] (b, B) \]

\[ \div (\pi y)^{-\frac{1}{2}} (4xy)^{\frac{1}{2}} H_{r+1, s+1}^{m, n+1} \left[ (u^2 4xy)^{\frac{1}{2}} \right] \left[ (\beta, \delta), (a, A), (\beta + 2v - 1, \delta) \right] (b, B) \].

(7)

**Proof:** The Laplace transform of $H$-function is given by\(^6 \text{[p. 140, Eqn. (2.4)]}\)

\[ \int_{0}^{\infty} e^{-t \beta} H_{r, s}^{m, n} \left[ (ut)^{\delta} \right] (a, A) (b, B) dt \]

\[ = p^{-\frac{1}{2}} H_{r+1, s+1}^{m, n+1} \left[ \left( \frac{u}{p} \right)^{\delta} \right] (\beta, \delta), (a, A) (b, B), \]

(8)

valid for $\text{Re}(p) > 0, \delta > 0, \lambda > 0, |\text{arg } u| < \lambda \pi/2, \text{Re}[-\beta + \delta (b_i/B_i)] > -1 (j = 1, 2, \ldots, m)$. On writing $(pq)^{-\frac{1}{2}}$ for $p$, multiplying both sides of (8) by $(pq)^{\frac{1}{2} - \frac{v}{2}} p^{-\frac{1}{2}}$ and then interpreting it with the help of the known result\(^3 \text{[p. 144 (3.26)]}, we get

\[ (\pi y)^{-\frac{1}{2}} (4xy)^{\frac{1}{2}} \left[ t \left( t - 2 \beta - 2v \right) \right] J_{2v-1} \left[ (64 xy t^2)^{\frac{1}{2}} \right] \]

\[ \times H_{r, s}^{m, n} \left[ (ut)^{\delta} \right] (a, A) (b, B) dt \]

\[ \div p^{-\frac{1}{2}} (pq)^{\frac{(3/2) - 1}{2}} H_{r+1, s+1}^{m, n+1} \left[ (u^2 pq)^{\frac{1}{2}} \right] \left[ (\beta, \delta), (a, A) \right] (b, B) \],

(9)

provided $\text{Re}(v) > 0$.

Now evaluating the left hand side integral with the help of a known formula\(^4 \text{[p. 326 (2)]}, we obtain the desired result valid under the conditions (i)-(iv) stated with the theorem.

3. **Particular cases**

By taking proper choice of the parameters in (7) and on using the known results\(^5 \text{(p. 54-68)}, we obtain the following two class of results:

(A) *The named image functions expressed in terms of the $H$-function*

For the sake of brevity we shall use the following abbreviations in this section:

\[ X = (pq)^{\frac{1}{2}}, \ Y = (4xy)^{\frac{1}{2}} \text{ and } \theta = (p\delta^2)^{-\frac{1}{2}}. \]
\[
\theta X^{6-4r} I_r (X) K_\mu (X) \\
= \frac{Y^{(4r-3)/2}}{4 \pi y^4} H^{\frac{\alpha}{2}} \left[ Y^{-\delta} \left( \left( \frac{1}{2} + v + \frac{1}{2} \right), \left( \frac{1}{2} - \frac{y}{4} + \frac{1}{2} \right) \right) \right].
\]

\[
\theta X^{5-4r} H^{\nu \nu} (X) H^{10} (X) \\
= \frac{(\cos v \pi)}{2 \pi^3 y^4} H^{\frac{\alpha}{2}} \left[ Y^{-\delta} \left( \left( \frac{1}{2} + \nu, \delta \right), \left( \frac{1}{2} - \frac{y}{4} + \frac{1}{2} \right) \right) \right].
\]

\[
\theta \chi \frac{X^{6-4r}}{2} M_{k,m} (2X) W_{k,m} (2X) \\
= \frac{\Gamma \left( 1 + 2m \right)}{2 \pi y^4} \left[ Y^{-\delta} \left( \left( \frac{1}{2} + m, \delta \right), \left( \frac{1}{2} + \nu + 1, \delta \right) \right) \right].
\]

\[
\theta X^{4-4r} W_{k,m} (X^2) \\
= \frac{Y^{1/4}}{\Gamma (1 + k + \frac{1}{2})} H^{\frac{\alpha}{2}} \left[ Y^{-\delta} \left( \left( \frac{1}{2} + k, \delta \right), \left( \frac{1}{2} + \nu + 1, \delta \right) \right) \right].
\]

\[
\theta X^{4-4r} \left[ I_\alpha (X) I_\alpha (X) - I_{-\alpha} (X) I_{-\alpha} (X) \right] \\
= \frac{-5i \pi (\mu + \nu) \pi}{\pi^3 y^4} Y^{(4x-3)/2} H^{\frac{\alpha}{2}} \left[ Y^{-\delta} \left( \left( \frac{1}{2} + \mu, \delta \right), \left( \frac{1}{2} - \frac{y}{4} + \frac{1}{2} \right) \right) \right].
\]

\[
\theta X^{4-4r} e^{-4X^2} I_\alpha (\frac{1}{2} X^2) \\
= \frac{Y^{(4x-1)/2}}{\pi y^4} H^{\frac{\alpha}{2}} \left[ Y^{-\delta} \left( \left( \frac{1}{2} + \nu + 1, \delta \right), \left( \frac{1}{2} + \nu + 1, \delta \right) \right) \right].
\]

\[
\theta \chi \frac{X^{5-4r}}{2} \left[ I_{-\alpha} (2X) - I_{-\alpha} (2X) \right] \\
= \frac{\cos (2x \pi)}{\pi^3 y^4} \left[ Y^{(4x-3)/2} H^{\frac{\alpha}{2}} \left( \left( \frac{1}{2} + v, \delta \right) \right) \right].
\]

\[
\theta X^{5-4r \nu} e^{4X^2} K_{\alpha} (\frac{1}{2} X^2) \\
= \frac{\sec (2x \pi)}{\pi y^4} \left[ Y^{(4x-1)/2} H^{\frac{\alpha}{2}} \left( \left( \frac{1}{2} + v, \delta \right) \right) \right].
\]

\[
\theta X^{4-4r \nu} W_{k,m} (2iX) W_{k,m} (-2iX) \\
= \frac{Y^{(4x-1)/2}}{\pi y^4} \left[ Y^{-\delta} \left( \left( \frac{1}{2} + m, \delta \right), \left( \frac{1}{2} + \nu + 1, \delta \right) \right) \right].
\]
BIVARIATE LAPLACE TRANSFORMS FOR SOME H-FUNCTIONS

\( b X^5 (1 - v) \left[ H_v (2X) - Y_v (2X) \right] \)

\[
\frac{\cos (v \pi)}{\pi^{b/2} y^b} y^{(5a - 2)_{1/2}} H^{\frac{1}{3}, 1}_{2, 1} \left[ y^{-\delta} \begin{pmatrix} \frac{V}{2} + 1, \delta, \left( \frac{5}{2} v, \delta \right) \\ \frac{V}{2} + 1, \delta, \left( \frac{1}{2} \pm \frac{v}{2}, \delta \right) \end{pmatrix} \right].
\]

\( \theta X^5 - 4 \mu S_\mu, v (2X) \)

\[
\frac{-y^{(4 + \mu - 2)_{1/2}} (\pi^2)^{1/2}}{2^{-\mu + 1} \Gamma \left( \frac{1}{2} - \frac{\mu}{2} \pm \frac{v}{2} \right)} H^{\frac{3}{2}, 1}_{2, 1} \left[ y^{-\delta} \begin{pmatrix} \frac{1}{2} + \mu, \delta, \left( 2 + \mu, \delta \right) \\ \left( \frac{1}{2} + \mu, \delta, \left( \frac{1}{2} \pm \frac{v}{2}, \delta \right) \end{pmatrix} \right].
\]

(B) The H-function expressed as a named original function

For the sake of brevity we shall use the following abbreviations in this section:

\[ Z = (4xy)^{-1}, \quad U = (pq)^{1/2}, \quad \phi = \left( \frac{p}{\delta^2} \right)^{-1}. \]

\[ \phi U^{(4 - w)_{1/2}} H^{\frac{1}{4}, 1}_{4, 1} \left[ y^{-\delta} \begin{pmatrix} \frac{1}{2} + \frac{1}{2} w, \delta \\ \frac{1}{2} + \frac{1}{2} w, \delta \end{pmatrix} \right].
\]

\[ \frac{Z^2 [J_\mu (Z) J_v (Z) - J_{-v} (Z) J_{-\mu} (Z)] y^{-\frac{1}{2}} \sec \left( \frac{\mu + v}{2} \right)}{\pi}. \]

\[ \phi U^{(4 - w)_{1/2}} H^{\frac{1}{4}, 1}_{4, 1} \left[ y^{-\delta} \begin{pmatrix} \frac{1}{2} + \frac{1}{2} w, \delta \\ \frac{1}{2} + \frac{1}{2} w, \delta \end{pmatrix} \right].
\]

\[ \frac{(\pi Z)^2 y^{-\frac{1}{2}}}{2} [(\cos \mu \pi + \cos v \pi)^{-1}] \times [e^{i \pi (v - \mu)_{1/2}} H^{(1)}_{v, \mu} (Z) + e^{i \pi (v - \mu)_{1/2}} H^{(1)}_{\mu, v} (Z) H^{(1)}_{v, \mu} (Z)].
\]

\[ \phi U^{(2 k - 2)_{1/2}} H^{\frac{1}{4}, 1}_{4, 1} \left[ y^{-\delta} \begin{pmatrix} \frac{U}{A} \\ \frac{a}{2} + \frac{1}{2} \pm m, \delta, \left( \frac{k + 1 + \frac{1}{2} a, \delta \right) \\ \frac{a}{2} + \frac{1}{2}, \delta, \left( \frac{a}{2} + \frac{1}{2}, \delta \right) \end{pmatrix} \right].
\]

\[ \frac{2^{1 - a} Z^{2k - 4} y^{-\frac{1}{2}} \Gamma \left( \frac{1}{2} - k \pm m \right) W_{k, m} (iZ) W_{k, m} (-iZ)}{2^{1 - a} Z^{2k - 4} y^{-\frac{1}{2}} \Gamma \left( \frac{1}{2} - k \pm m \right) W_{k, m} (iZ) W_{k, m} (-iZ)}.
\]
\[
\phi U^{(4 m + 1)} \int \frac{\Gamma(m + k, \delta)}{\Gamma(m + \frac{1}{2}, \delta)} \left( \begin{array}{c}
\frac{1}{4} + k, \delta \\
\frac{1}{4} + k, \delta
\end{array} \right)
\]

\[\Gamma(m - k + \frac{1}{2}) Z^{2m} \left[ \Gamma(2m + 1) \right]^{-1} y^{1-m} W_{\frac{1}{2}, m}^{(2Z)} M_{\frac{1}{2}, m}^{(2Z)} \]

4. Acknowledgements

Thanks are due to Dr. Sampooran Singh, Director, Defence Laboratory, Jodhpur for giving encouragement and to Dr. S. Krishna and Dr. P. C. Munot for taking keen interest in the work. I am also thankful to the referee for giving certain suggestions for improvement of the paper.

References


