Short Communication

Some results on the independence number of a graph

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Abstract
In this paper, we give new lower bounds for the independence number \( \alpha(G) \) of a finite and simple graph \( G \).

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Graphs, considered here, are finite and simple (without loops or multiple edges), and [1], [2] are followed for terminology and notation. Let \( G = (V, E) \) be an undirected graph, with the set of vertices \( V = \{ v_1, v_2, \ldots, v_n \} \) and the set of edges \( E \), such that \( |E| = m \).

We denote by \( d(v) \) the degree of a vertex \( v \) in \( G \). It is well known (e.g. see [2]) that \( \alpha(G) = \sum_{i=1}^{n} d(v_i) = 2m \).

Let \( \delta_i(v) \) be the number of vertices having the distance \( i \) from a vertex \( v \) of \( G \) and let \( \alpha(G) \) be the independence number of \( G \).

**LEMMA 1.** If \( G \) is a triangle-free graph, then
\[
\alpha(G) \geq \alpha^*(G) = \sum_{v \in V} \delta_1(v)/(1 + \delta_1(v) + \delta_2(v)).
\]

**Proof.** We randomly label the vertices of \( G \) with a permutation of the integers from 1 to \( n \). Let \( S \subseteq V \) be the set of vertices \( v \) for which the minimum label on vertices at distance 0, 1 or 2 from \( v \) is on a vertex at distance 1. Obviously, the probability that \( S \) contains a vertex \( v \) is given by \( \delta_1(v)/(1 + \delta_1(v) + \delta_2(v)) \) and, therefore, the expected size of \( S \) is equal to \( \alpha^*(G) \). Moreover, \( S \) must be an independent set of \( G \), since, otherwise, if \( S \) contains an edge it is easy to see that it must lie in a triangle of \( G \), contradicting the hypothesis. Thus, the lemma is proved.

**THEOREM 1.** If \( G \) is a triangle- and pentagon-free graph with \( m \) edges, then \( \alpha(G) \geq \sqrt{m} \).

**Proof.** Let \( d(G) \) be the average degree of vertices of \( G \). Since \( G \) is a triangle- and pentagon-free graph, then we have \( \alpha(G) \geq \delta_1(v) \), by considering the neighbours of \( v \), and \( \alpha(G) \geq 1 + \delta_2(v) \), by considering \( v \) and the vertices at distance 2 from \( v \), for any vertex \( v \) of \( G \). Thus, by the above lemma, \( \alpha(G) \geq \alpha^*(G) \geq \sum_{v \in V} \delta_1(v)/2 \alpha(G) \), that is,
\[
\alpha(G)^2 \geq nd(G)/2 \text{ or } \alpha(G) \geq \sqrt{nd(G)/2}.
\]
But, \( d(G) \geq \sigma(G)/n = 2m/n \) and, therefore, \( \alpha(G) \geq \sqrt{m} \), the theorem being proved.

**LEMMA 2.** If \( G \) is a graph with an odd girth \( 2k+3 \) (\( k \geq 2 \)) or greater, then
\[
\alpha(G) \geq \sum_{v \in V} (\frac{1}{2} (1 + \delta_1(v) + \ldots + \delta_{k-1}(v)))/ (1 + \delta_1(v) + \ldots + \delta_k(v)).
\]
Proof. We randomly label the vertices of $G$ with a permutation of the integers from 1 to $n$. Let $S_1 \subseteq V$ (respectively $S_2 \subseteq V$) be the set of vertices $v$ for which the minimum label on vertices at distance $k$ or less from $v$ is at even (respectively odd) distance $k - 1$ or less. It is easy to see that $S_1$ and $S_2$ are independent sets and that the expected size of $S_1 \cup S_2$ is given by

$$\sum_{v \in V} (1 + \delta_1(v) + \ldots + \delta_{k-1}(v))/(1 + \delta_1(v) + \ldots + \delta_k(v)),$$

the lemma being proved.

THEOREM 2. If $G$ is a graph with an odd girth $2k + 3$ ($k \geq 2$) or greater, then $\alpha(G) \geq 2^{-(k - 1)/k} (\sum_{v \in V} \delta_1(v)^{1/(k-1)})^{(k - 1)/k}$.

Proof. By Lemma 1 and applying Lemma 2 for all the values between 3 and $k$, we have,

$$\alpha(G) \geq \sum_{v \in V} \delta_1(v)/(1 + \delta_1(v) + \delta_2(v)) + \frac{1}{2} ((1 + \delta_1(v) + \delta_2(v))/(1 + \delta_1(v) + \delta_2(v) + \delta_3(v))) + \ldots + \frac{1}{2} ((1 + \delta_1(v) + \ldots + \delta_{k-1}(v))/(1 + \delta_1(v) + \ldots + \delta_k(v)))/(k - 1).$$

Since the arithmetic mean is greater than the geometric mean, then

$$\alpha(G) \geq \sum_{v \in V} ((\delta_1(v) 2^{-(k - 2)})/(1 + \delta_1(v) + \ldots + \delta_k(v)))^{1/(k - 1)}.$$ 

Since the vertices at even (odd) distance less than or equal to $k$ from any vertex $v$ of $G$ form independent sets, then

$$2\alpha(G) \geq 1 + \delta_1(v) + \ldots + \delta_k(v).$$

Thus,

$$\alpha(G) \geq \sum_{v \in V} (\delta_1(v)/2^{k-1} \alpha(G))^{1/(k-1)}$$

or

$$\alpha(G)^{k/(k-1)} \geq \frac{1}{2} (\sum_{v \in V} \delta_1(v)^{1/(k-1)})$$

or

$$\alpha(G) \geq 2^{-(k - 1)/k} (\sum_{v \in V} \delta_1(v)^{1/(k-1)})^{(k - 1)/k},$$

the theorem being proved.

COROLLARY. If $G$ is a regular graph of the degree $r(G)$ and with an odd girth $2k + 3$ ($k \geq 2$) or greater, then

$$\alpha(G) \geq 2^{-(k - 1)/k} n^{(k - 1)/k} r(G)^{1/k}.$$ 

Proof. It follows, immediately, from Theorem 2.

Remark. Marcu [3] presents an algorithm with a computer program which for a given graph $G$ finds all its maximal independent sets and the exact value of $\alpha(G)$. 

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References