Steering Control of Semilinear Discrete Dynamical System

Raju K George¹ AND Trupti P Shah²

Abstract | In this paper, we investigate the controllability property of a class of semilinear non-autonomous system described by the difference equation

\[ x(t+1) = A(t)x(t) + B(t)u(t) + f(t, x(t)), \quad t \in N_0 = \{0, 1, 2, \ldots\} \]

under the assumption that its linear part is controllable and the nonlinear function \( f \) satisfies Lipschitz condition. We also give an algorithm to compute steering control for the above system. Numerical example is given to illustrate the result.

1. Introduction
In [1], Krabs studied the controllability of a general difference system of the form

\[ x(t+1) = f(x(t), u(t)). \]

Further, they have also obtained a controller that steers a given initial state to a desired final state for the linear system (1.2). In this paper we consider a semi-linear system of difference equation of the form

\[ x(t+1) = A(t)x(t) + B(t)u(t) + f(t, x(t)), \]

\[ x(0) = x_0, \quad t \in N_0 \quad (1.1) \]

and the corresponding linear system:

\[ x(t+1) = A(t)x(t) + B(t)u(t), \]

\[ x(0) = x_0, \quad t \in N_0. \quad (1.2) \]

Here, \( (A(t))_{t \in N_0} \) and \( (B(t))_{t \in N_0} \) are sequences of real \( n \times n \) and \( n \times m \) matrices, respectively, and \( (x(t))_{t \in N_0} \) and \( (u(t))_{t \in N_0} \) are sequences of state vectors in \( R^n \) and control vectors in \( R^m \), respectively, \( f(\ldots) : N_0 \times R^n \to R^n \) is a nonlinear function satisfying Lipschitz condition with respect to the second argument.

We introduce a steering controller for system (1.1) and prove that it is well-defined and it steers any initial state \( x_0 \) of system (1.1) to a desired final state \( x_1 \) in \( N \in N_0 \) time steps under certain conditions.

We define the problem of controllability and reachability as follows.

Problem of Controllability
Let \( x_0, x_1 \in R^n \) be given arbitrarily. We say that the system is controllable if there exists a sequence of control vectors \( (u(t))_{t \in N_0}, \quad t \in N_0 \), such that for some \( N \in N_0 \) the solution \( (x(t))_{t \in N_0} \) of equation (1.1) starting from the initial state \( x(0) = 0 \), also satisfies the end condition \( x(N) = x_1 \).

Problem of Reachability
We say that the state \( x_1 \in R^n \) is reachable in \( N \) time steps, if there exist a sequence of control vectors \( u(t) \in R^m, \quad t \in N_0 \), such that the corresponding solution starting from \( x(0) = 0 \), also satisfies \( x(N) = x_1 \).

We now express the solution of (1.1) and (1.2) in terms of the state-transition matrix \( \Phi(t, t_0) \)
associated with the homogeneous linear part of (1.2).

The state transition matrix $\Phi(t, t_0)$ is given by [2]

$$\Phi(t, t_0) = A(t-1)A(t-2)\ldots A(t_0) \quad \forall \ t \geq t_0$$

It can be shown that the solution of (1.1) is given by

$$x(t) = \Phi(t, 0)x_0 + \sum_{j=0}^{t-1} \Phi(t, j+1)B(j)u(j)$$

and the solution of (1.2) is given by

$$x(t) = \Phi(t, 0)x_0 + \sum_{j=0}^{t-1} \Phi(t, j+1)f(j, x(j))$$

In this article, we give the computational scheme for the steering control. For $t = 0, 1, \ldots, N-1$, we define a controller

$$u(t) := B(t)^* \Phi(N, t + 1)^* W_r(0, N)^{-1}$$

$$\begin{bmatrix} x_1 - \Phi(N, 0)x_0 \\ - \sum_{j=0}^{N-1} \Phi(N, j+1)f(j, x(j)) \end{bmatrix}$$

for the nonlinear system (1.1), where $W_r(0, N)$ is called reachability Grammian defined by

$$W_r(0, N) := $$

$$\sum_{j=0}^{N-1} \Phi(N, j+1)B(j)B(j)^* \Phi(N, j+1)^*$$

We will prove that this control is well-defined and steers the nonlinear system (1.1) from $x_0$ to $x_1$. We make the following assumptions to obtain the result.

**Assumptions**

[LI]: The linear system (1.2) is controllable.

[N]: The nonlinear function $f(t, x)$ is Lipschitz continuous with respect to $x$. That is, there exists $\alpha > 0$ such that

$$\| f(t, x) - f(t, y) \| \leq \alpha \| x - y \| \quad \forall x, y \in \mathbb{R}^n$$

Under the above assumptions, we prove in Section 2 that the system (1.1) is controllable and also prove that controllability and reachability of the system (1.1) are equivalent. Numerical example for steering control of system (1.1) is provided in Section 3.

Further, let $S_N = S_N(R^n)$ ($N \geq 0$) be the linear space of terminating sequences $\{x(t)\}_{t=0}^N$, $x(t) \in \mathbb{R}^n$ and denote by $S_\infty^N = S_\infty^N(R^n)$, the corresponding Banach space with norm $\| \cdot \|_N^\infty :

$$\| x \|_N^\infty = \sup_{0 \leq t \leq N} \| x(t) \|$$

We denote the linear space of control sequences by

$$U_{[0, N]} = \{ u \in \mathbb{R}^{n(N+1)} :$$

$$u := [u(0), u(1), \ldots, u(N)],$$

with $u(t) \in \mathbb{R}^n$, $0 \leq t \leq N$$

The following propositions will be employed to prove our results.

**Proposition 1.1.** (Callier and Desoer [3]). Let $(A(t)), (B(t)), t \in N_0$ be given compatible matrix-sequences. Then the following are equivalent:

(i) The linear system (1.2) is controllable on $[0, N]$.

(ii) $\det W_r(0, N) \neq 0$, where the reachability grammian $W_r$ is as defined in (1.6).

**Proposition 1.2.** If the system (1.2) is controllable on $[0, N]$, then for all $x_0, x_1 \in \mathbb{R}^n$, there exists $u \in U_{[0, N]}$ defined by

$$u(t) := B(t)^* \Phi(N, t + 1)^* W_r(0, N)^{-1}$$

$$\times [x_1 - \Phi(N, 0)x_0]$$

that steers the initial state $x_0$ to the desired final state $x_1$ in $N$ time-steps.

**2. Main Results**

**Theorem 2.1.** If the linear system is controllable in $N$ time-steps and the control $u(t)$ defined by (1.5) is well-defined, then it steers the nonlinear system (1.1) from the initial state $x_0$ to the desired final state $x_1$ in $N$ time-steps.

**Proof:** Since the linear system (1.2) is controllable on $[0, N]$, we have by Proposition 1.1 that $\det W_r(0, N) \neq 0$. If we substitute the control given by (1.5) in the solution

$$x(t) = \Phi(t, 0)x_0 + \sum_{j=0}^{t-1} \Phi(t, j+1)B(j)u(j)$$

$$+ \sum_{j=0}^{t-1} \Phi(t, j+1)f(j, x(j))$$
we get,

$$x(t) = \Phi(t, 0)x_0 + \sum_{j=0}^{t-1} \Phi(t, j+1)B(j)B(j)^* \times \Phi(N, j+1)^*W_r(0, N)^{-1} \times \Phi(N, 0)x_0$$

$$- \sum_{j=0}^{N-1} \Phi(N, j+1)f(j, x(j))$$

$$+ \sum_{j=0}^{t-1} \Phi(t, j+1)f(j, x(j)) \quad (2.1)$$

It can be easily verified that at $t = 0$, $x(0) = x_0$ and at $t = N$, $x(N) = x_1$. Thus, the control $u$ defined in (1.5) steers the non-linear system from the given initial state $x_0$ to the desired final state $x_1$.

We now prove that the control defined in (1.5) is meaningful. This control $u$ is well-defined if there is a solution to the equation (2.1) with this control. We will prove existence and uniqueness of solution of (2.1).

We make use of the following notations and definitions:

Let $C = \max_{N \geq t \geq 0} \| \Phi(t, j) \|$, $M_1 = \max_{N \geq j \geq 0} \| B(j) \|$ and $M_2 = \| W_r(0, N)^{-1} \| \cdot 

\beta = C(1 + C^2M_1^2M_2(N - 1)) 

\eta = \alpha \beta (N - 1)$

**Theorem 2.2.** Under Assumptions [L],[N] and $\eta < 1$ the steering control defined by

$$u(t) = B(t)^* \Phi(N, t+1)^*W_r(0, N)^{-1} \times \left[ x_1 - \Phi(N, 0)x_0 - \sum_{j=0}^{N-1} \Phi(N, j+1)f(j, x(j)) \right]$$

is well-defined.

**Proof:** We prove this by showing that the nonlinear system with this control has a unique solution. In Theorem 2.1 we have shown that this control does the required steering. We show that the following nonlinear equation has a unique solution.

$$x(t) = \Phi(t, 0)x_0 + \sum_{j=0}^{t-1} \Phi(t, j+1)B(j)B(t)^* \times \Phi(N, t+1)^*W_r(0, N)^{-1} \times \Phi(N, j+1)^*W_r(0, N)^{-1} \times \Phi(N, 0)x_0$$

$$- \sum_{j=0}^{N-1} \Phi(N, j+1)f(j, x(j))$$

$$+ \sum_{j=0}^{t-1} \Phi(t, j+1)f(j, x(j))$$

To prove the existence of the solution, we define a mapping

$$T : S_N^\infty(R^n) \rightarrow S_N^\infty(R^n)$$

by

$$T(x(t)) = \Phi(t, 0)x_0 + \sum_{j=0}^{t-1} \Phi(t, j+1)B(j)B(t)^* \times \Phi(N, t+1)^*W_r(0, N)^{-1} \times \Phi(N, j+1)^*W_r(0, N)^{-1} \times \Phi(N, 0)x_0$$

$$- \sum_{j=0}^{N-1} \Phi(N, j+1)f(j, x(j))$$

$$+ \sum_{j=0}^{t-1} \Phi(t, j+1)f(j, x(j))$$

Since $S_N^\infty(R^n)$ is a complete Banach space, we show that operator $T$ has a fixed point by using Banach Contraction mapping theorem.

Consider

$$\| T(x(t)) - T(\tilde{x}(t)) \| \leq \sum_{j=0}^{t-1} \| \Phi(t, j+1)B(j)B(t)^* \times \Phi(N, t+1)^*W_r(0, N)^{-1} \times \Phi(N, j+1)^*W_r(0, N)^{-1} \times \Phi(N, 0)x_0$$

$$- \sum_{j=0}^{N-1} \Phi(N, j+1)f(j, x(j))$$

$$+ \sum_{j=0}^{t-1} \| \Phi(t, j+1)f(j, x(j)) \|$$

$$\leq C \sum_{j=0}^{t-1} \| f(x(j)) - f(x(j)) \|$$

$$+ C^2M_1^2M_2 \sum_{j=0}^{t-1} \sum_{i=0}^{N-1} \| f(i, x(i)) - f(i, \tilde{x}(i)) \| + C^2M_1^2M_2 \sum_{j=0}^{t-1} \sum_{i=0}^{N-1} \| f(i, \tilde{x}(i)) - f(i, x(i)) \|$$

We have shown that $C \beta$ is a contraction mapping, hence $T$ has a unique fixed point $\tilde{x}(t)$. Therefore, the control $u(t)$ defined in (1.5) steers the system from $x_0$ to $x_1$.
\[ \leq \alpha C \sum_{j=0}^{t-1} \| x(j) - \tilde{x}(j) \| \]
\[ + \alpha C^3 M_1^2 M_2 (t-1) \sum_{i=0}^{N-1} \| \tilde{x}(i) - x(i) \| \]
\[ \leq \alpha C \sum_{j=0}^{N-1} \| x(j) - \tilde{x}(j) \| \]
\[ \leq \alpha \beta \sum_{j=0}^{N-1} \| x(j) - \tilde{x}(j) \|. \]

Thus, \[ \sup_{0 \leq t \leq N} \| T(x(t)) - T(\tilde{x}(t)) \| \leq \alpha \beta (N-1) \sup_{0 \leq t \leq N} \| x(t) - \tilde{x}(t) \| \]
\[ \| T(x) - T(\tilde{x}) \| \leq \eta \| x - \tilde{x} \|. \]

Since \( \eta < 1 \), \( T \) is a contraction. Hence \( T \) has a unique fixed point. Therefore, the nonlinear equation is uniquely solvable. This proves that the control defined in (1.5) is well-defined.

We now give the following computational result for the steering control for the nonlinear system.

**Theorem 2.3.** Under the assumptions of Theorem 2.2, the steering control and controlled trajectory of the nonlinear system (1.1) driving the system from \( x(0) = x_0 \) to \( x(N) = x_1 \) can be computed by the following iterative scheme:

\[
u^m(t) = B(t)^* \Phi(N, t+1)^* W_r(0, N)^{-1} \]
\[ \times \left[ x_1 - \Phi(N, 0) x_0 \right. \]
\[ \left. - \sum_{j=0}^{N-1} \Phi(N, j+1) f(j, x^m(j)) \right] \tag{2.2} \]

and

\[
x^{m+1}(t) = \Phi(t, 0) x_0 + \sum_{j=0}^{t-1} \Phi(t, j+1) B(j) u^m(j) \]
\[ + \sum_{j=0}^{t-1} \Phi(t, j+1) f(j, x^m(j)) \tag{2.4} \]

starting with arbitrary \( x^0(t), t = 0, 1, 2, \ldots, N-1, \]
\[ m = 0, 1, 2, \ldots \]

**Proof:** The computational scheme follows directly from Banach contraction principle and from Theorem 2.2.

Although for nonlinear systems controllability and reachability notions are not equivalent, we prove in the following theorem that for the semilinear system (1.1), the two notions are equivalent.

**Theorem 2.4.** The two notions of controllability and reachability are equivalent for the semilinear system (1.1).

**Proof:** From definition, it is obvious that for the system (1.1), controllability implies reachability. Conversely, let the system (1.1) is reachable on \([0, N]\). Thus, the 0 state can be steered to any desired state \( \tilde{x}_1 \).

Now, for arbitrary \( x_0, x_1 \in R^n \), choose \( \tilde{x}_1 = x_1 - \Phi(N, 0) x_0 \).

Since there exists \( u(t) \in U_{[0, N]} \) that steers \( x_0 = 0 \) to \( \tilde{x}_1 \) for some \( N \). Hence

\[
\tilde{x}_1 = \Phi(N, 0) x_0 + \sum_{j=0}^{N-1} \Phi(N, j+1) B(j) u(j) + \sum_{j=0}^{N-1} \Phi(N, j+1) f(j, x(j)) \tag{2.5}
\]

i.e.

\[
x_1 = \Phi(N, 0) x_0 + \sum_{j=0}^{N-1} \Phi(N, j+1) B(j) u(j) + \sum_{j=0}^{N-1} \Phi(N, j+1) f(j, x(j)) \tag{2.6}
\]

3. Numerical Example

Example 1. Consider the nonlinear system given by the following equation:

\[
x(t+1) = A(t)x(t) + B(t)u(t) + f(t, x(t)) \tag{3.1}
\]
Steering Control of Semilinear Discrete Dynamical System

Figure 1: Controlled trajectories.

![Controlled trajectories diagram]

where \( A(t) = \frac{1}{4} \begin{pmatrix} \cos(2t) & 1 \\ t^2 & \cos^2(t) \end{pmatrix} \),

\( B(t) = \begin{pmatrix} .5 \\ .5t \end{pmatrix} \) and

\( f(t, x) = \frac{1}{2} \begin{pmatrix} \sin^2(x_1(t)) \\ \cos^2(x_2(t)) \end{pmatrix} \)

Let \( N = 10 \). Here the reachability Grammian can be computed as

\[ W_r(0, 10) = \begin{pmatrix} 1.0774 & 0.2760 \\ 0.2760 & 0.2078 \end{pmatrix} \]

and \( \det W_r(0, N) = 0.1477 \neq 0 \).

Hence the linear system is controllable, and

\[
\| f(t, x) - f(t, y) \| \\
\leq \frac{1}{2} \left( \sin^2(x_1) - \sin^2(y_1) \right) \\
\leq \frac{2}{5} \| x - y \|
\]

Hence \( f \) is Lipschitz with Lipschitz constant \( \frac{2}{5} \). We can easily verify the conditions of Theorem 2.2 to conclude that the system is controllable. Figure 1 shows the controlled trajectory steering the system from the initial state \( x_0 = \begin{pmatrix} 4 \\ -4 \end{pmatrix} \) to the final state \( x_1 = \begin{pmatrix} -4 \\ 4 \end{pmatrix} \).

Received 19 November 2007; revised 28 November 2007.

References


Raju K George has obtained PhD from Indian Institute of Technology, Bombay in 1991 and was National Board of Higher Mathematics (NBHM) Post-doctoral fellow at Indian Institute of Science, Bangalore during 1991–1994. His research work includes controllability analysis of nonlinear systems using functional analytic techniques, Neural Networks, optimal control problems etc. He worked at the M S University of Baroda for 12 years as Associate Professor. He was a visiting faculty at the University of Delaware, Delaware, USA during 2002–2004. He has also worked at the University Institute of Chemical Technology (formerly UDCT), Bombay as Professor of Mathematics in 2007. Currently he is Professor of Mathematics at Indian Institute of Space Science and Technology (IIST), Trivandrum. He has 28 research publications in national and international journals. He is a reviewer in many journals and Mathematical Reviews.

Mrs. Trupti P. Shah received the M.Sc. (Mathematics) and P.G. Diploma in Computer Science and Application (P.G.D.C.A.) from the Sardar Patel University, Vallabh Vidyanagar during year 1992 and 1993 respectively. Then she worked as a Part time lecturer in Arts, Science and R. A. Patel Commerce college, Bhadran and Bhavan’s college, Dakor for one year. She joined dept. of Applied Mathematics, M.S. University of Baroda, Vadodara in 1994 and currently working as a lecturer in the same department. She has 13 years teaching experience in Applied Mathematics and Computer programming. She is currently doing research in the field of controllability and stability problems of discrete time systems. Her interests include study of optimal control problems and boundary value problems of nonlinear discrete dynamical systems and Numerical methods.