Characterisations of compact operators on the space of almost periodic functions

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Abstract

Let $X$ be a Banach space and $AP(R, X)$ the space of continuous almost periodic functions on $R$ into $X$ with supremum norm. We obtain characterisations of compact operators on $AP(R, X)$ (Theorem 3.1). This theorem is then used to prove the following: Let $AP(R)$ be the space $AP(R, \mathbb{C})$. For a compact operator $K$ on $AP(R)$, the aggregate of Fourier exponents of functions in the range of $K$ is countable even though the range of $K$ is uncountable. We also obtain sufficient conditions on $K$ so that the Fourier series of all the functions in the range of $K$ converges at the same point.

Keywords: Compact operators, almost periodic functions.

1. Introduction

It is well known that a bounded linear operator on $AP(R)$, the space of continuous complex-valued almost periodic functions on $R$, is compact if and only if it can be approximated by a trigonometric polynomial on $AP(R)$\(^{1}\). This appears to be the only known characterisation of compact operators on $AP(R)$. This result vitally depends on the fact that $AP(R)$ is a Banach algebra with pointwise product and supremum norm. In fact, the proof uses the Gelfand theory. If $AP(R, X)$ denote the set of continuous almost periodic functions on $R$ into a Banach space $X$, then $AP(R, X)$ is a Banach space with supremum norm. Since $AP(R, X)$ is not a Banach algebra, in general, the techniques of Schaeffer\(^{1}\) are not applicable to the operators on $AP(R, X)$. However, we show that, it is possible to obtain a similar characterisation of compact operators on $AP(R, X)$, by using elementary properties of almost periodic functions. We prove that an operator $K$ on $AP(R, X)$ is compact if and only if it is approximated by an operator-valued trigonometric polynomial on $AP(R, X)$. In Theorem 3.1 some more characterisations are obtained.

For Banach spaces $X$ and $Y$, let $BL(X, Y)$ denote the space of bounded linear operators on $X$ into $Y$ with uniform operator topology. We denote by $KL(X, Y)$, the subspace of $BL(X, Y)$ consisting of compact operators. When $X = Y$ we write $BL(X)$ and $KL(X)$ for $BL(X, Y)$ and $KL(X, Y)$, respectively. Let $A = AP(R, X)$. For $K$ in $BL(A)$ and $t \in \mathbb{R}$, define
Let $K_r: A \rightarrow A$ by $K_r f = (Kf) _ r$, where $f \in A$ and $(Kf) _ r$ is the translate of $Kf$ by $r$. It is proved that (Theorem 3.1) an operator $K$ on $A$ is compact if and only if the map $F: \mathbb{R} \rightarrow KL(A, X)$ defined by $F(t)(f) = Kf(t), t \in \mathbb{R}, f \in A$, is continuous almost periodic. We further show that this is equivalent to the fact that the function $\theta^K: t \mapsto K_r$, from $\mathbb{R}$ into $KL(A, X)$ is continuous almost periodic. In Corollary 3.2, it is proved that a compact operator $K$ on $AP(\mathbb{R}, X)$ can be represented by an operator-valued almost periodic function up to an isometric isomorphism.

When $X = \mathbb{C}$, we write $AP(\mathbb{R})$ for the space $AP(\mathbb{R}, X)$. Let $B$ be the unit ball in $AP(\mathbb{R})$. If $K$ is a compact operator on $AP(\mathbb{R})$, we obtain in Section 4, sufficient conditions on the map $\theta^K: t \mapsto K_r$, from $\mathbb{R}$ into $KL(AP(\mathbb{R}))$, so that the Fourier series of $Kf$, for all $f \in AP(\mathbb{R})$, converge at the same point. Also, for a compact operator $K$ on $AP(\mathbb{R})$, we use the results of Section 3 to prove the following: The union of sets of Fourier exponents of functions in the range of $K$ is countable, even though the range of $K$ contains uncountably many functions. In fact, we show that this set is contained in the set of Fourier exponents of $\theta^K$.

2. Preliminaries

The following definitions are from Corduneanu\textsuperscript{2} and Burckel\textsuperscript{3}:

**Definition 2.1.** Let $X$ be a Banach space. A continuous function $f: \mathbb{R} \rightarrow X$ is called almost periodic, if for any number $\epsilon > 0$, one can find a number $l(\epsilon) > 0$ such that any interval of the real line of length $l(\epsilon)$ contains at least one point of abscissa $\tau$ with the property that $\| f(t + \tau) - f(t) \| < \epsilon$ for all $t \in \mathbb{R}$.

**Definition 2.2.** A set $S$ is called a topological semigroup if $S$ is a semigroup with identity $e$ and if $S$ has a Hausdorff topology such that the multiplication on $S$ is separately continuous. That is, for each $t$ in $S$, the maps $s \mapsto st$ and $s \mapsto ts$ are continuous functions.

Let $S$ be a topological semigroup, $X$, a Banach space and $C(S, X)$ the Banach space of bounded continuous functions from $S$ to $X$ with supremum norm. For $f$ in $C(S, X)$ and $s$ in $S$, let $f_s$, the right translate of $f$ by $s$, be defined by $f_s (t) = f(ts), t \in S$. Let the right orbit of $f$, $O_R (f) = \{ f_s ; s \in S \}$.

**Definition 2.3.** A function $f$ in $C(S, X)$ is called almost periodic if $O_R(f)$ is relatively compact in $C(S, X)$.

We shall denote by $AP(S, X)$, the set of all almost periodic functions on $S$ to $X$. When $X = \mathbb{C}$, we write $AP(S)$ for $AP(S, \mathbb{C})$. It is easy to see that $AP(S, X)$ is a Banach space with the norm defined by $\| f \| = \sup_{s \in S} \| f(s) \|$. The space $AP(S, X)$ has been studied by Goldberg and Irwin\textsuperscript{4}. When $S = \mathbb{R}$, the equivalence of definition 2.1 and definition 2.3 is proved in by Corduneanu\textsuperscript{2} (Theorem 6.6).

**Definition 2.4.** A function $T: \mathbb{R} \rightarrow X$ defined by

$$T(t) = \sum_{k=1}^{\infty} c_k e^{iat}, \quad t \in \mathbb{R},$$
where, for $1 \leq k \leq n$, $\lambda_k$ are real numbers and $c_k$ are in $X$, is called a trigonometric polynomial with values in $X$.

**Definition 2.5.** (Approximation property). A function $f: \mathbb{R} \to X$ is called a function with the approximation property, if for any $\varepsilon > 0$, one can determine a trigonometric polynomial $T_\varepsilon$ with values in $X$, such that $\|f(t) - T_\varepsilon(t)\| < \varepsilon, t \in \mathbb{R}$.

**Remark 1.** It is proved$^5$ [Theorem 2.11] that $f: \mathbb{R} \to X$ is almost periodic if and only if it has the approximation property.

**Definition 2.6.** A family $\mathcal{F}$ in $AP(\mathbb{R}, X)$ is said to be equalalmost periodic, if to any $\varepsilon > 0$, there corresponds a number $l(\varepsilon) > 0$, such that any interval of length $l(\varepsilon)$ contains at least one number $t$ for which $\|f(t + \tau) - f(t)\| < \varepsilon$ for all $f \in \mathcal{F}$ and for all $t \in \mathbb{R}$.

The following theorem is from Corduneanu$^2$ [Theorem 6.9].

**Theorem 2.7.** A finite family of functions in $AP(\mathbb{R}, X)$ is equalalmost periodic.

For general theory of almost periodic functions on $\mathbb{R}$ with values in a Banach space $X$, we refer to Corduneanu$^2$ and Levitan & Zhikov$^7$. It may be recalled that, for Banach space $X$, $Y$, an operator $K$ in $BL(X, Y)$ is defined to be compact if the set $\{Kx: \|x\| \leq 1\}$ is relatively compact in $Y$.

### 3. Characterisations of compact linear operators on $AP(\mathbb{R}, X)$

Throughout this section, $X$ is a Banach space and $A$ stands for the space $AP(\mathbb{R}, X)$. We shall obtain here some characterisations of compact linear operators on $A$ in terms of operator-valued trigonometric polynomials and translates of operators defined in introduction. The proof of the following uses only the elementary properties of almost periodic functions in $A$.

**Theorem 3.1.** Let $K \in BL(A)$. Then the following are equivalent:

(i) $K$ is compact

(ii) The map $F: \mathbb{R} \to KL(A, X)$ defined by $F(t)(f) = Kf(t), f \in A, t \in \mathbb{R}$, is continuous almost periodic.

(iii) For each $\varepsilon > 0$, there exists a trigonometric polynomial $T_\varepsilon$ in $KL(A)$ such that $\|K - T_\varepsilon\| < \varepsilon$.

(iv) The map $\theta^K: \mathbb{R} \to KL(A)$ defined by $\theta^K(t) = K_\tau, t \in \mathbb{R}$ is continuous almost periodic.

**Proof.** (i) $\Rightarrow$ (ii). Assume that $K$ is compact. Let $B$ be the closed unit ball in $A$ and $\varepsilon > 0$.

Since $KB$ is relatively compact in $A$, we can choose a finite set $\{f^{(1)}, \ldots, f^{(n)}\}$ in $B$ so that $\{Kf^{(i)}: 1 \leq i \leq n\}$ is an $\varepsilon/3$-net for $KB$. Given $f \in B$, choose $i \in \{1, \ldots, n\}$ so that $\|Kf - Kf^{(i)}\| < \varepsilon/3$. Since the family $\{Kf^{(i)}: 1 \leq i \leq n\}$ is uniformly equicontinuous, there exists $\delta > 0$ such that, whenever $|s - t| < \delta$, $\|Kf^{(i)}(s) - Kf^{(i)}(t)\| < \varepsilon/3$ for all $i = 1, \ldots, n$. Hence, whenever
\[ |s - t| < \delta, \text{ we have} \]
\[ \| F(s)(f) - F(t)(f) \| = \| Kf(s) - Kf(t) \| \]
\[ \leq \| Kf(s) - Kf(t) \| + \| Kf(t)(s) - Kf(t)(t) \| \]
\[ \leq \| Kf - Kf(t) \| + \| Kf(t)(s) - Kf(t)(t) \| \]
\[ \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \]

Therefore, \[ \| F(s) - F(t) \| = \sup_{s \in \mathbb{R}} \| F(s)(f) - F(t)(f) \| \leq \varepsilon, \] whenever \[ |s - t| < \delta. \] This proves the uniform continuity of \[ F. \]

Now to prove the almost periodicity of \[ F, \] note that as above
\[ \| F(s + \tau)(f) - F(s)(f) \| \leq \| Kf - Kf(t) \| + \| Kf(t)(s + \tau) - Kf(t)(s) \| + \| Kf(t) - Kf(t) \|. \]
It is then clear that the equialmost periodicity of \[ \{ Kf(t) : 1 \leq i \leq n \} \] implies \[ F \] is almost periodic.

(ii) \implies (iii). If \[ F \] is continuous almost periodic, then \[ F \] has the approximation property (Section 2, Remark 1). Therefore, for each \[ \varepsilon > 0, \] there exists a trigonometric polynomial \[ P_\varepsilon \in AP(\mathbb{R}, KL(A, X)) \] such that \[ \| F - P_\varepsilon \| < \varepsilon, \] where \[ P_\varepsilon(t) = \sum_{k=1}^{n} a_k e^{-i\lambda_k t}, \] \[ a_k \in KL(A, X), \] \[ \lambda_k \in \mathbb{R}, \] \[ 1 \leq k \leq n. \] Now, define \[ T_\varepsilon : A \rightarrow A \] by \[ T_\varepsilon f(t) = P_\varepsilon(t)(f). \] We first prove that \[ T_\varepsilon \] is compact. Let \[ B \] be the unit ball in \[ A. \] Since \[ P_\varepsilon \] is almost periodic \[ \text{[Theorem 6.5]}, \] the set \[ \{ P_\varepsilon(t) : t \in \mathbb{R} \} \] is relatively compact in \[ KL(A, X). \] Let \[ \{ P_\varepsilon(t) : 1 \leq i \leq m \} \] be a finite \[ \varepsilon/3 \]-net for \[ \{ P_\varepsilon(t) : t \in \mathbb{R} \}. \] Also since \[ a_k \] are compact operators, the terms \[ a_k e^{-i\lambda_k t} \] are compact. As \[ P_\varepsilon(t) \] is a finite sum of such terms, it is compact. In particular, \[ P_\varepsilon(t) \] is compact for each \[ i = 1, \ldots, m. \] Now if \[ H : A \rightarrow X^m \] is defined by \[ Hf = (P_\varepsilon(t^1)(f), \ldots, P_\varepsilon(t^m)(f)), \] then it is easy to see that \[ HB \] is relatively compact in \[ X^m. \] Let \[ Hf^1, \ldots, Hf^m \] be a finite \[ \varepsilon/3 \] - net for \[ HB. \] Then for any \[ f \in B, \] there exists \[ f^0 \text{ such that } \| Hf - Hf^0 \|_X^m < \varepsilon/3. \] But

\[ \| Hf - Hf^0 \|_X^m = \sum_{i=1}^{m} \| P_\varepsilon(t^i)(f) - P_\varepsilon(t^i)(f^0) \|. \]

Hence

\[ \| P_\varepsilon(t^i)(f) - P_\varepsilon(t^i)(f^0) \| < \varepsilon/3 \text{ for all } i = 1, \ldots, n \] \hfill (I)

Let \[ t \in \mathbb{R} \text{ and } f \in B. \] Choose \[ i \in \{ 1, \ldots, n \} \] such that \[ \| P_\varepsilon(t) - P_\varepsilon(t^0) \| < \varepsilon/3 \] and then \[ j \in \{ 1, \ldots, n \} \] so that (I) holds. Then

\[ \| T_\varepsilon f(t) - T_\varepsilon f^0(t) \| = \| P_\varepsilon(t)(f) - P_\varepsilon(t^0)(f^0) \| \]
\[ \leq \| P_\varepsilon(t)(f) - P_\varepsilon(t^0)(f) \| \]
\[ + \| P_\varepsilon(t^0)(f) - P_\varepsilon(t^0)(f^0) \| \]
\[ + \| P_\varepsilon(t^0)(f^0) - P_\varepsilon(t)(f^0) \| \]
\[ < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \]
Therefore,
\[ \| T_\epsilon f - T_\epsilon f^{(i)} \| = \sup_{t \in \mathbb{R}} \| T_\epsilon f(t) - T_\epsilon f^{(i)}(t) \| \leq \epsilon. \]
This shows that \( \{ T_\epsilon f^{(i)} : 1 \leq i \leq n \} \) is a finite \( \epsilon \)-net for \( T_\epsilon \). Hence \( T_\epsilon \) is compact. Finally, since
\[ \| K - T_\epsilon \| = \sup_{\| f \| \leq 1} \| Kf - T_\epsilon f \| \leq \sup_{\| f \| \leq 1} \sup_{t \in \mathbb{R}} \| Kf(t) - T_\epsilon f(t) \| \]
\[ = \sup_{\| f \| \leq 1} \sup_{t \in \mathbb{R}} \| F(t)(f) - P_\epsilon(t)(f) \| = \sup_{\| f \| \leq 1} \| F(t) - P_\epsilon(t) \| = \| F - P_\epsilon \|, \]
assertion (iii) follows.

(iii) \( \Rightarrow \) (i) If (iii) holds then \( K \) is compact, since it is a uniform limit of compact operators \( T_\epsilon \).

(ii) \( \Rightarrow \) (iv). Suppose that \( F \) is continuous almost periodic and let \( \epsilon > 0 \). Since \( F \) is uniformly continuous on \( \mathbb{R}^2 \) [Theorem 6.2], there exists \( \delta > 0 \) such that, whenever \( |s - t| < \delta \),
\[ \| F(x + s) - F(x + t) \| < \epsilon \]
for all \( x \in \mathbb{R} \). But as
\[ \sup_{x \in \mathbb{R}} \| F(x + s) - F(x + t) \| = \sup_{\| f \| \leq 1} \| F(x + s)(f) - F(x + t)(f) \| \]
\[ = \sup_{\| f \| \leq 1} \| Kf(x + s) - Kf(x + t) \| \]
\[ = \sup_{\| f \| \leq 1} \| (Kf)_s - (Kf)_t \| \]
\[ = \sup_{\| f \| \leq 1} \| K_s(f) - K_t(f) \| \]
\[ = \| K_s - K_t \| \]
\[ = \| \theta^K(s) - \theta^K(t) \|, \]
it follows that \( \theta^K \) is continuous. Also, from the above equalities we have, for any \( t \),
\[ \| \theta^K(t + \tau) - \theta^K(t) \| = \sup_{x \in \mathbb{R}} \| F(x + t + \tau) - F(x + t) \| \]
\[ = \sup_{x' = x + \tau \in \mathbb{R}} \| F(x' + \tau) - F(x') \|. \]

Hence, the almost periodicity of \( \theta^K \) follows from that of \( F \). (iv) \( \Rightarrow \) (ii). From the equalities in (ii) \( \Rightarrow \) (iv), it is clear that the continuity and almost periodicity of \( F \) follow from that of \( \theta^K \).

Remark 1. The proof of the above theorem vitally depends on the fact that the functions in \( AP(\mathbb{R}, X) \) have approximation property. If \( G \) is a locally compact abelian topological group then the algebra of trigonometric polynomials in \( G \), that is, the algebra of finite linear combinations of the continuous characters on \( G \), is norm dense in \( AP(G) \) [Larsen\(^7\), Theorem 10.7.4]. More generally, if \( S \) is a topological semigroup and algebraically an abelian group then by Burcke\(^3\), Corollary 5.6, the space of finite linear combinations of semicharacters of \( S \) is norm dense in \( AP(S) \). Hence in the above theorem, when \( X = \mathbb{C} \), we can replace \( \mathbb{R} \) by either a locally compact abelian group or a locally compact topological semigroup which is algebraically an abelian group. The exponential functions are then replaced by continuous characters or semicharacters.
Corollary 3.2. $KL(A)$, the Banach space of compact operators is isometrically isomorphic to a subspace of $AP(\mathbb{R}, KL(A))$.

Proof. From the above theorem, if $K$ is compact then the map $\theta^K$ defined by $\theta^K(t) = K_t$ is continuous almost periodic. This defines a map $\psi: KL(A) \rightarrow \theta^K$ from $KL(A)$ into $AP(\mathbb{R}, KL(A))$. It is easy to see that $\psi$ is a linear homomorphism. Also as $\| (Kf)_s \| = \| Kf \|$ for any $s \in \mathbb{R}$ we have

$$
\| \psi(K) \| = \| \theta^K \| = \sup_{s \in \mathbb{R}} \| \theta^K(s) \| = \sup_{s \in \mathbb{R}} \| K_s \| = \sup_{s \in \mathbb{R}} \| K_s f \| = \sup_{s \in \mathbb{R}} \| (Kf)_s \| = \sup_{t \in \mathbb{R}} \| Kf \| = \| K \|
$$

which shows that $\psi$ is an isometry. This completes the proof.

Remark 2. When $X = \mathbb{C}$, Theorem 3.1 can be generalised to obtain characterisations of collectively compact sets of operators on $A = AP(\mathbb{R})$. It may be recalled that for Banach spaces $X$, $Y$ a set $\mathcal{F} \subseteq BL(X, Y)$ is collectively compact if $\{Kx: \| x \| \leq 1, K \in \mathcal{F} \}$ is relatively compact in $Y^\circ$. Let $A^\circ$ denote the topological dual of $AP(\mathbb{R})$. For each $K \in BL(A)$ define $F^K: \mathbb{R} \rightarrow A^\circ$ by $t \rightarrow F^K(t)$, where $F^K(t)(f) = Kf(t)$, $f \in A$, and $\theta^K: \mathbb{R} \rightarrow BL(A)$ by $\theta^K(t) = K_t$, $t \in \mathbb{R}$. Theorem 3.1 then can be generalised as follows:

Theorem 3.3. For a set of operators $\mathcal{F}$ in $BL(A)$, the following are equivalent:

(i) $\mathcal{F}$ is collectively compact.

(ii) The family $\{F^K: K \in \mathcal{F}\}$ is uniformly equicontinuous and equialmost periodic.

(iii) The family $\{\theta^K: K \in \mathcal{F}\}$ is uniformly equicontinuous and equialmost periodic.

4. Fourier series

Let $K$ be a compact operator on $AP(\mathbb{R})$ and $\theta^K$, as in Theorem 3.1. In this section, we apply the results of Section 3 to obtain sufficient conditions on the map $\theta^K$ so that the Fourier series of $Kf$, for all $f$ in $A$, converge at the same point. We also investigate the relation between the Fourier exponents of functions in the range of $K$ with those of $\theta^K$. We show that even though the range of $K$ is uncountable the union of sets of Fourier exponents of functions in the range of $K$ is countable and in fact this set is contained in the set of Fourier exponents of $\theta^K$.

Throughout this section, $A$ denotes the space $AP(\mathbb{R})$. To obtain the desired results, we first defined vector-valued functions of bounded variation in a way suggested by scalar functions of bounded variation.

Definition 4.1. Let $f$ be a function defined on the interval $[a, b]$ in $\mathbb{R}$ with values in Banach space $X$. If $P$ is the partition of $[a, b]$ given by $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, put $V(P, f) =$
\[ \sum_{i=1}^{n} \| f(t_i) - f(t_{i-1}) \|. \] Then the function \( f \) is said to be of bounded variation on \([a, b]\) if and only if \( \sup_{P} V(P, f) < \infty \).

The following theorem is from Corduneanu\textsuperscript{2} [Theorem 1.21].

**Theorem 4.2.** Assume that the almost periodic function \( f(x) \) is such that \( \lambda_{n+1} - \lambda_n \geq \alpha > 0 \), \( n = 1, 2, \ldots \), so that the unique limit point of its Fourier exponents is the point at infinity. If \( x_0 \) is a point in the neighbourhood of which \( f(x) \) has bounded variation, then the Fourier series at \( x_0 \) converges to \( f(x_0) \).

**Theorem 4.3.** Let \( K \in KL(A) \) and \( B \) the closed unit ball in \( A \). If \( \theta^K \) is as in Theorem 3.1 and satisfies

(i) \( \theta^K \) has bounded variation in the neighbourhood of a point (say) \( t_0 \) in \( \mathbb{R} \).

(ii) The Fourier exponents of \( \theta^K \) are such that \( \lambda_{n+1} - \lambda_n \geq \alpha > 0 \), \( n = 1, 2, \ldots \) (That is, the unique limit point of the Fourier exponents of \( \theta^K \) is the point at infinity).

Then, (a) The Fourier exponents of \( Kf \), for all \( f \in A \), belong to the set of Fourier exponents of \( \theta^K \).

(b) For all \( f \in A \), the Fourier series of \( Kf \) converges at the point \( 2t_0 \).

**Proof.** Since \( \theta^K \) is of bounded variation in the neighbourhood, say \([a, b]\), of \( t_0 \),

\[ \sup_{P} \sum_{i=1}^{n} \| \theta^K(t_i) - \theta^K(t_{i-1}) \| < \infty, \]

where supremum is taken over all partition \( P \) of \([a, b]\). But

\[ \| \theta^K(t_i) - \theta^K(t_{i-1}) \| = \| K_{t_i} f - K_{t_{i-1}} f \| \]

\[ = \sup_{f \in B} \sup_{t \in \mathbb{R}} | K\gamma(t) f - K\gamma(t - 1) f | \]

\[ \geq | K\gamma(t_0 + t) - K\gamma(t_0 + t - 1) |, \]

for all \( f \in B \). Now, if \( 0 \neq f \in A \) is arbitrary then \((f/\| f \|) \in B\) and we have

\[ \| \theta^K(t_i) - \theta^K(t_{i-1}) \| \geq \frac{1}{\| f \|} \| K\gamma(t_0 + t) - K\gamma(t_0 + t - 1) \|. \]

Therefore, if \( u_i = t_0 + t_i, \quad 1 \leq i \leq n, \)

\[ \sup_{P} \sum_{i=1}^{n} \| K\gamma(u_i) - K\gamma(u_{i-1}) \| \leq \left( \sup_{P} \sum_{i=1}^{n} \| \theta^K(t_i) - \theta^K(t_{i-1}) \| \right) \| f \| < \infty. \]

This shows that for all \( f \in A \), \( K\gamma \) is of bounded variation in \([a + t_0, b + t_0]\), a neighbourhood of \( 2t_0 \). Now let \( \sum_{k=1}^{\infty} A_k e^{i\lambda_k t} \) be the Fourier series associated with \( \theta^K \), where

\[ A_k = a(\lambda_k, \theta^K) = \lim_{n \to \infty} \frac{1}{n} \int_{0}^{\pi} \theta^K(t) e^{-it\lambda_k} dt. \]
Then
\[ \|a(\lambda, \theta^K)\| = \sup_{\|f\| \leq 1} \|a(\lambda, \theta^K)(f)\| \]
\[ = \sup_{\|f\| \leq 1} \sup_{x \in \mathbb{R}} |a(\lambda, \theta^K)(f)(x)| \]
\[ = \sup_{\|f\| \leq 1} \sup_{x \in \mathbb{R}} \left| \lim_{n \to \infty} \int_{0}^{n} \theta^K(t)(f)e^{-i\lambda s} dt \right| \]
\[ = \sup_{\|f\| \leq 1} \sup_{x \in \mathbb{R}} \left| \lim_{n \to \infty} \int_{0}^{n} Kf(x)e^{-i\lambda s} dt \right| \]
\[ = \sup_{\|f\| \leq 1} \sup_{x \in \mathbb{R}} \left| \lim_{n \to \infty} \int_{0}^{n} Kf(x+t)e^{-i\lambda s} dt \right| \]
\[ \geq \left| \lim_{n \to \infty} \int_{0}^{n} Kf(x+t)e^{-i\lambda s} dt \right| \text{ for all } f \in B, x \in \mathbb{R}. \]

Changing \( t \) to \( t - x \), we have
\[ \text{RHS} = \left| \lim_{n \to \infty} \int_{-\infty}^{n} Kf(t)e^{-i\lambda (t-x)} dt \right| \]
\[ = \left| \lim_{n \to \infty} \int_{0}^{n} Kf(t)e^{-i\lambda t} dt \right| \text{ by Corduneanu}^{2}; \text{ Theorem 1.12 and} \]
\[ \text{since } |e^{-i\lambda x}| = 1. \]

Hence, \( \text{RHS} = |a(\lambda, Kf)| \), where \( a(\lambda, Kf) \) is the Fourier coefficient of \( Kf \). Thus \( \|a(\lambda, \theta^K)\| \geq \|a(\lambda, Kf)\| \) for all \( f \in B \). Now, if \( 0 \neq f \in A \) be any element then \( (f/\|f\|) \in B \) and we have \( \|a(\lambda, \theta^K)\| \geq (1/\|f\|) \|a(\lambda, Kf)\| \). Therefore, for any \( f \in A \), if \( a(\lambda, Kf) \neq 0 \) then \( a(\lambda, \theta^K) \neq 0 \). This shows that, for all \( f \in A \) if \( \lambda \) is a Fourier exponent of \( Kf \), then it is a Fourier exponent of \( \theta^K \). In other words, the union of sets of Fourier exponents of functions in the range of \( K \) is contained in the set of Fourier exponents of \( \theta^K \). This proves (a). Therefore, if \( \theta^K \) satisfies (ii) so does \( Kf \), for all \( f \in A \). The assertion (b) then follows from Theorem 4.2.

We now give an example of a compact operator \( K \) on \( A \) and estimate the set of Fourier exponents of \( \theta^K \).

**Example.** For \( f \in A \) define \( K:A \to A \) by
\[ Kg(s) = \lim_{n \to \infty} \int_{0}^{n} f(s-t)g(t) dt. \]

Since \( f \) is almost periodic, from Theorem 3.1 it can be easily proved that \( K \) is compact. Let \( \theta^K \) be as in Theorem 3.1. We shall show that the set of Fourier exponents of \( \theta^K \) is precisely the set of Fourier exponents of \( f \). Let
\[ a(\lambda, \theta^K) = \lim_{m \to \infty} \int_{0}^{m} \theta^K(s)e^{-i\lambda s} ds. \]
Therefore,

\[ \text{RHS} = \sup_{\|g\| \leq 1} \sup_{x \in \mathbb{R}} \left| \lim_{m \to \infty} \frac{1}{m} \int_{0}^{m} \left( \lim_{n \to \infty} \frac{1}{n} \int_{0}^{n} f(x + s - t)g(t) \, dt \right) e^{-i_ns} \, ds \right| \]

\[ = \sup_{\|g\| \leq 1} \sup_{x \in \mathbb{R}} \left| \lim_{m \to \infty} \frac{1}{m} \int_{0}^{m} \left( \lim_{n \to \infty} \phi_{n}(x) \right) e^{-i_ns} \, ds \right| \]  

where

\[ \phi_{n}(x) = \frac{1}{n} \int_{0}^{n} f(x + s - t)g(t) e^{-i_{ns}} \, dt. \]

We shall show that \( \phi_{n}(x) \) converges uniformly in \( s \). It is enough to prove that \( \phi_{n}(x) \) is Cauchy uniformly in \( s \). Since \( f \) is almost periodic, there exists \( s_{1}, \ldots, s_{k} \) in \( \mathbb{R} \) such that for any \( s \in \mathbb{R} \), there is \( s_{i}, 1 \leq i \leq k \) with

\[ \| f_{s} - f_{s_{i}} \| \leq \frac{\varepsilon}{3 \| g \|}. \]

Let \( x \in \mathbb{R} \). For each \( i = 1, \ldots, k \) define \( g_{i}(t) = f(x + s_{i} - t)g(t) \). Then \( g_{i}, 1 \leq i \leq k \), is almost periodic function. Therefore

\[ \lim_{n \to \infty} \frac{1}{n} \int_{0}^{n} g_{i}(t) \, dt \]

exists for each \( i \). Hence there exists \( N(i) \) such that

\[ \left| \frac{1}{n} \int_{0}^{n} g_{i}(t) \, dt - \frac{1}{n'} \int_{0}^{n'} g_{i}(t) \, dt \right| < \frac{\varepsilon}{3} \]

for all \( n, n' \geq N(i) \). Let \( N = \max \left\{ N(i) : 1 \leq i \leq k \right\} \). Then, for \( n, n' \geq N \) we have

\[ \left| \frac{1}{n} \int_{0}^{n} f(x + s_{i} - t)g(t) \, dt - \frac{1}{n'} \int_{0}^{n'} f(x + s_{i} - t)g(t) \, dt \right| < \frac{\varepsilon}{3}. \]
for all $i \leq k$. Now, for $n, n' \geq N$ and $s \in \mathbb{R}$,

$$|\phi_n(s) - \phi_n'(s)| = \left| \frac{1}{n} \int_0^n f(s + x - t)g(t)e^{-i\lambda s} \, dt - \frac{1}{n'} \int_0^{n'} f(s + x - t)g(t)e^{-i\lambda s} \, dt \right|$$

$$\leq \frac{1}{n} \left| \int_0^n \left[ f(s + x - t) - f(s_i + x - t) \right]g(t) \, dt \right|$$

$$+ \frac{1}{n'} \left| \int_0^{n'} f(s + x - t)g(t) \, dt - \int_0^{n'} f(s_i + x - t)g(t) \, dt \right|$$

$$+ \frac{1}{n'} \left| \int_0^{n'} \left[ f(s_i + x - t) - f(s + x - t) \right]g(t) \, dt \right|$$

$$\leq \left\| f_s - f_s' \right\| \left\| g \right\| + \frac{e}{3} + \left\| f_{s_i} - f_s \right\| \left\| g \right\|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus

$$|\phi_n(s) - \phi_n'(s)| < \varepsilon \quad \text{for all } n, n' \geq N \text{ and for all } s \in \mathbb{R}.$$  \hspace{1cm} (II)

Hence, $\lim_{n \to \infty} \phi_n(s)$ exists uniformly for all $s \in \mathbb{R}$. But then from (I) we have

$$\left\| a(\lambda, \theta^k) \right\| = \sup_{\| \theta \| \leq 1} \sup_{x \in \mathbb{R}} \left| \lim_{m \to \infty} \frac{1}{m} \int_0^m \phi_n(s) \, ds \right|.$$  \hspace{1cm} (III)

Let

$$a_n^m = \frac{1}{m} \int_0^m \phi_n(s) \, ds.$$

Write $a_n = \lim_{n \to \infty} a_n^m$ and $a_n = \lim_{n \to \infty} a_n^m$. We shall prove that $a_n^m$ converges to $a_n^m$ uniformly in $m$. But again, it is enough to show that $a_n^m$ is uniformly cauchy in $m$. From (II) if $n, n' \geq N$, we have

$$|a_n^m - a_n^m| = \left| \frac{1}{m} \int_0^m \phi_n(s) \, ds - \frac{1}{m} \int_0^m \phi_n'(s) \, ds \right|$$

$$\leq \frac{1}{m} \int_0^m |\phi_n(s) - \phi_n'(s)| \, ds < \varepsilon.$$

This shows that $\lim_{n \to \infty} a_n^m$ exists uniformly in $m$. Therefore, the sequence $\{a_n\}$ converges and $\lim_{m \to \infty} a_n^m = \lim_{n \to \infty} a_n$. Equivalently,

$$\lim_{m \to \infty} \lim_{n \to \infty} a_n^m = \lim_{n \to \infty} \lim_{m \to \infty} a_n^m.$$

Hence, from (II),

$$\left\| a(\lambda, \theta^k) \right\| = \sup_{\| \theta \| \leq 1} \sup_{x \in \mathbb{R}} \left| \lim_{m \to \infty} \lim_{n \to \infty} \frac{1}{m} \int_0^m \phi_n(s) \, ds \right|.$$
COMPACT OPERATORS ON THE SPACE OF A.P. FUNCTIONS

\[ \sup_{|g| \leq 1} \sup_{x \in \mathbb{R}} \left| \lim_{n \to \infty} \lim_{m \to \infty} \frac{1}{m} \int_{0}^{m} \left[ \frac{1}{n} \int_{0}^{n} f(x + s - t)g(t)e^{-itx} dt \right] ds \right| = \sup_{|g| \leq 1} \sup_{x \in \mathbb{R}} \left| \lim_{n \to \infty} \lim_{m \to \infty} \frac{1}{m} \int_{0}^{m} f(x + s - t)e^{-itx} ds \right| g(t) dt \]

If \( \psi_m(t) = \frac{1}{m} \int_{0}^{m} f(x + s - t)g(t)e^{-itx} ds \), then it can be shown that \( \psi_m(t) \) converges uniformly in \( t \). Therefore, taking limit inside the integral and changing \( s \) to \( s + t \) we have

\[ \text{RHS} = \sup_{|g| \leq 1} \sup_{x \in \mathbb{R}} \left| \lim_{n \to \infty} \lim_{m \to \infty} \frac{1}{m} \int_{0}^{m} f(x + s)e^{-itx} ds \right| g(t)e^{-itx} dt \]

\[ = \sup_{|g| \leq 1} \sup_{x \in \mathbb{R}} \left| \lim_{n \to \infty} \lim_{m \to \infty} \frac{1}{m} \int_{0}^{m} f(x + s)e^{-itx} ds \right| g(t)e^{-itx} dt \]

\[ = \sup_{|g| \leq 1} \sup_{x \in \mathbb{R}} |a(\lambda, f)| \left| \lim_{n \to \infty} \frac{1}{n} \int_{0}^{n} g(t)e^{-itx} dt \right| = |a(\lambda, f)|, \] since \( |a(\lambda, f)| = |a(\lambda, g)| \)

\[ = |a(\lambda, f)|, \] since \( \sup_{|g| \leq 1} |a(\lambda, g)| = 1. \]

Thus, \( \|a(\lambda, g^k)\| = |a(\lambda, f)| \), which shows that \( \lambda \) is a Fourier exponent of \( g^k \) if and only if it is a Fourier exponent of \( f \).

References